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ABSTRACT

This book is part 1 of a 2-part manual for teachers using SMSG text materials for grade 6. The purpose for each of 5 chapters is stated and mathematical background for the teacher is presented. Detailed lesson plans are then provided, including sequences of statements and questions, activities, and exercise sets with answers. Needed materials and vocabulary are listed. Chapter topics include: (1) exponents; (2) multiplication of rational numbers; (3) side and angle relationships of triangles; (4) integers; and (5) coordinates. (MN)

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Mathematics for the Elementary School, Grade 6

Teacher's Commentary, Part I

REVISED EDITION

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

As one of its contributions to the improvement of mathematics in the schools of this country, the School Mathematics Study Group has prepared a series of sample text materials for grades 4 through 6. These are designed to illustrate a kind of mathematics curriculum that we believe appropriate for elementary schools.

This volume is a portion of these materials which were prepared by a group of 30 individuals, divided almost equally between distinguished college and university mathematicians and master elementary teachers and consultants. A strong effort has been made on the part of all to make the content of this text material mathematically sound, appropriate and teachable. Preliminary versions were used in numerous classrooms both to strengthen and to modify these judgments.

The content is designed to give the pupil a much broader concept, than has been traditionally given at this level, of what mathematics really is. There is less emphasis on rote learning and more emphasis on the construction of models and symbolic representation of ideas and relationships from which pupils can draw important mathematical generalizations.

The basic content is aimed at the development of some of the fundamental concepts of mathematics. These include ideas about: number; numeration; the operations of arithmetic; and intuitive geometry. The simplest treatment of these ideas is introduced early. They are frequently re-examined at each succeeding level and opportunities are provided throughout the texts to explore them more fully and apply them effectively in solving problems. These basic mathematical understandings and skills are continually developed and extended throughout the entire mathematics curriculum, from grades K through 12 and beyond.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that these texts may greatly assist all pupils and teachers who use them to achieve this goal, and that they may experience something of the joy of discovery and accomplishment that can be realized through the study of mathematics.

Chapter 1

EXPONENTS

PURPOSE OF UNIT

To extend the pupils' understanding of factoring and of the unique factorization theorem for whole numbers.

To provide experiences in the use of exponents to write new names for numbers.

To strengthen pupils' understanding of place value in the decimal system of numeration for whole numbers.

To illustrate the advantages of the exponent form in naming very large numbers.

To introduce the techniques for multiplication and division of numbers expressed as powers of a common base.

MATHEMATICAL BACKGROUND

Numbers and Numerals

Confusion frequently exists regarding the terms number and numeral. These are not synonymous. A number is a concept, an abstraction. A numeral is a symbol, a name for a number. A numeration system is a numeral system (not a number system); it is a system for naming numbers.

Admittedly, there are times when making the distinction between "number" and "numeral" becomes somewhat cumbersome. In contexts where only one numeral system is being used or where the emphasis is on number rather than numeral, it is convenient to make the customary identification of a number with its name. When, however, numerals are themselves the objects of study it is essential to distinguish a number from its names and its names from one another. Since this chapter is concerned with a new type of numeral, an attempt has been made to preserve such distinctions.

The Meaning of Equality

It is important to understand clearly the correct way in which the equals sign (=) is to be used.

For example, when we write

$$4 \times 3 = 24 \div 2$$

we are asserting that the symbols "4×3" and "$24 \div 2$" are each names for the same thing--the number 12. In general, when we write

$$A = B$$

we do not mean that the letters or symbols "A" and "B" are the same. They very evidently are not! What we do mean is that the letters "A" and "B" are being used as synonyms.

That is, the equality

$$A = B$$

asserts precisely that the thing named by the symbol "A" is identical with the thing named by the symbol "B". The equals sign always should be used only in this sense.

Exponents, Bases, and Powers

There are many instances in mathematics in which we use a certain number more than once as a factor. Examples are found in the computation of area of a square, $A = s \times s$; in the volume of a cube, $V = e \times e \times e$; and in the volume of a sphere, $V = \frac{4}{3} \times \pi \times r \times r \times r$.

Another illustration of the use of a number several times as a factor is found in the designation of place value in our numeration system. The numeral 1,486, for example, can be thought of as an abbreviation for

$$(1 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (8 \times 10) + 6$$

A numeral which names a number as a product is called a product expression. Thus $5 \times 2 \times 3$ is a product expression for the number with decimal numeral 30. If a product expression shows the same factor repeated there is a simple way to abbreviate it. For example the expression

$3 \times 3 \times 3 \times 3$ is abbreviated 3^4,

$2 \times 2 \times 2$ is abbreviated 2^3,

and $t \times t$ is abbreviated t^2.

In such abbreviations the numeral naming the repeated factor is called the base, and the raised numeral designating the number of repetitions is called the exponent. (Note: this use of the term "base" is related to but different from its use in place value numeral systems. The numeral 5^3 contains the decimal numeral 5 as base, but it is not a "base five" numeral. However, the form 5^3 can be directly translated into the base five numeral for this number, namely 1000 five.)

When used as a numeral, the symbol 3^4 is read "three to the fourth power". Since it names 81, we call the number 81 the *fourth power of three*. Similarly 2^3 is read "two to the third power" or "two cubed", and t^2 is read "t to the second power" or "t squared". We will call numerals like 3^4, 2^3, and t^2 *exponential forms* of the product expressions which they abbreviate. We will also say that they express numbers as powers. They are simple abbreviated forms of product expressions. Any expression containing product expressions which are abbreviated to exponential forms will also be called an *exponential form*. Thus

$$s^2,\ e^3,\ \frac{4}{3} \times \pi \times r^3, \text{ and}$$

$$(1 \times 10^3) + (4 \times 10^2) + (8 \times 10) + 6$$

are exponential forms.

Positional Notation in the Decimal System of Numeration

Our numeration system is called the decimal system because it uses groups of ten. The word decimal comes from the Latin word "decem" which means "ten." The decimal system is widely used throughout the world today.

It is probable that the reason a decimal numeral system evolved is that people have ten fingers. The ten symbols we use are called digits.

The decimal system uses the idea of place value to represent the size of a group. The size of the group represented by a digit depends upon the place or the position of the digit in a numeral. The digit indicates how many of that group there are. This clever idea of place value makes it possible to express a number of any size by the use of only ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Since grouping is by tens and powers of ten in the decimal system, its *base* is ten. Each successive place to the left indicates a place value ten times as large as that of the preceding place. Reading from the right, the first place indicates ones, the second place indicates tens, or ten times one

(10×1). The third place indicates ten times ten (10×10), or one hundred; the next, ten times ten times ten $(10 \times 10 \times 10)$ or one thousand, and so on.

A decimal numeral, such as 2345, may be interpreted as an abbreviation of a sum expression.

2345 means $(2 \times 1000) + (3 \times 100) + (4 \times 10) + (5 \times 1)$, or

2345 means $2000 + 300 + 40 + 5$.

When we write the numeral, 2345, we are using number symbols, the idea of place value, and base ten.

The decimal system has an advantage over the Roman numeral system in that it has a symbol for zero. The numeral 0 is used to fill places which would otherwise by empty. Without the use of some such numeral as 0 the situation would be more confusing. For example, the numeral for one thousand seven is 1007. Without a symbol for zero to mark the empty places, this might be confused with 17.

Expanded Notation

When the meaning of a number is expressed in the form shown below for 333, it is said to be written in expanded notation

$333 = (3 \times 100) + (3 \times 10) + (3 \times 1)$

Such an expression can be put in exponent form in the sense that 10, 100, 1000, etc. can be expressed as powers of 10.

$333 = (3 \times 10^2) + (3 \times 10^1) + (3 \times 1)$; or

$333 = (3 \times 10^2) + (3 \times 10) + (3 \times 1)$.

Since 5^1 means 5 and 10^1 means 10, the exponent 1 is written only for uniformity or for emphasis.

Products and Quotients of Numbers Written in Exponent Form

When numbers to be multiplied or divided can be expressed as powers of the same number, the operations may be performed by short and convenient procedures using only exponential forms.

The rules for these procedures may be discovered by pupils from the study of several examples. The general rule for multiplication is illustrated and stated below.

Give an exponential form for n if $n = 2^3 \times 2^4$

$$\boxed{\begin{aligned} &n = 2^3 \times 2^4 \\ &n = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \\ &n = 2^7 \end{aligned}}$$

It seems that

If $n = 2^3 \times 2^4$, then $n = 2^{(3 + 4)}$ or $n = 2^7$

The above example illustrates the rule that

$$y^b \times y^c = y^{(b + c)}$$

for counting numbers y, b and c

Let $n = y^b \times y^c$, then

$$n = \underbrace{(y \times y \times y \times \ldots \times y)}_{\text{b times}} \times \underbrace{(y \times y \times y \times \ldots \times y)}_{\text{c times}}$$

In the above expression, y is a factor $(b + c)$ times. So

$$y^b \times y^c = y^{(b + c)}$$

The general rule for division is illustrated and stated below.

Find an exponential form for n if $n = 10^5 \div 10^2$

$$\boxed{\begin{array}{ll} n = 10^5 \div 10^2 & \text{so} \\ n \times 10^2 = 10^5. & \text{But} \\ 10^3 \times 10^2 = 10^5. & \text{Therefore} \\ \qquad n = 10^3 & \end{array}}$$

This rule is a consequence of

(1) the relation between multiplication and division, and

(2) the rule for multiplying in exponential forms.

To say that $y^b \div y^c = y^{(b - c)}$ is to say that

$$y^c \times y^{(b - c)} = y^b.$$

We know however that

$$y^c \times y^{(b - c)} = y^{c + (b - c)} = y^b \text{ (multiplication rule).}$$

Thus we know that $y^b \div y^c = y^{(b - c)}$.

The value of these rules is this: if each of two numbers is given by an exponential form with the same base, there is a very simple way, involving only addition, to name their product in exponential form. Thus, to find the decimal numeral for 27×81 involves another complex procedure while an exponential form for $3^3 \times 3^4$ requires only the computation $3 + 4 = 7$. Of course the two approaches give two different numerals, namely 2,187 and 3^7. However, there are many instances in which 3^7 is to be preferred.

TEACHING THE UNIT

THE MEANING OF EXPONENT

Objective: To help pupils learn to write and interpret exponent forms.

Vocabulary: Factor, product, product expression, repeated factor, exponent, base, power, exponent form

Teaching Procedures:

In general, the exploration for each section of this unit is included in the pupils' book. Only a few further suggestions are included in this Teachers' Commentary.

While there is more than one way to teach this unit effectively, the following procedure is suggested as one possibility.

Much of the content, except the pages containing Exercise Sets, will be studied using questions which pupils can answer orally, but would have difficulty in answering in writing. At the same time there are many exercises included in these sections some of which can be completed by the pupils individually.

The Exercise Sets have been designed to be used as independent activities. Examples are in many cases supplied at the beginning of each set so the pupil can complete the exercises in acceptable form. A discussion of different methods used by pupils in solving these exercises is an excellent procedure.

The teacher will find that there is considerable reading expected of pupils. It is important that they learn to read text material with many numerals. Hence they should be asked to read certain sentences silently and afterwards discuss the contents of those pages.

Pupils in general learn to use exponents to express numerals quite readily. One difficulty they do encounter is the vocabulary. There are many new words in this section. All pupils should have many opportunities to use and illustrate the meaning of these words. For example on page 2 of the pupils' book, many more examples like exercise 1 should be used. For some pupils, even the word, factor, may be new.

Exercise Set 1 need not be completed as one assignment. Parts of it may be used after the prerequisite concepts have been emphasized.

P1

Chapter 1

EXPONENTS

THE MEANING OF EXPONENT

Suppose you were asked to read a very large number such as one that told you the distance to a star or one that gave the weight of the earth in pounds. These numbers and many others like them are so very large that you would have difficulty reading them. For example, the earth's weight is about 13,000,000,000,000,000,000,000,000 pounds.

This is a very large number. Can you read it? Can you think of some way in which you might tell a friend what the weight of the earth is in pounds?

In this chapter you will learn new ways of reading and writing these large numbers. These new ideas will be used often in mathematics and science courses which you will study later.

Product Expressions and Repeated Factors

1. The sentence 7 × 9 = 63 shows that 63 is the product of 7 and 9. It also shows that 7 and 9 are factors of 63. Because it names a number as a product, an expression like 7 × 9 is called a product expression. Give other product expressions for 63 if there are any.
(21 × 3) (1 × 63)

2. Write the decimal numeral for each of the following product expressions.

(a) 3 × 15 (45)
(b) 3 × 16 (48)
(c) 4 × 20 (80)
(d) 4 × 12 (48)
(e) 2 × 24 (48)
(f) 3 × 18 (54)
(g) 3 × 4 × 4 (48)
(h) 2 × 3 × 3 (18)
(i) 3 × 2 × 2 × 2 × 2 (48)
(j) 2 × 2 × 3 × 5 (60)
(k) 5 × 5 × 5 (125)
(l) 3 × 7 × 7 (147)

3. How many times is the factor 2 used in the product expression in (j) above? (2) In (i)? (4)

4. What factor is used more than once in example (h)? (3)

5. What number is shown as a repeated factor in example (l)? (7)

6. Write one or more product expressions for each of the following. Show at least one repeated factor in each product expression. The number of blanks will help you with some of them.

Example: 16 = 4 × 4
16 = 2 × 2 × 2 × 2

(a) 27 = (3) × (3) × (3)

(b) 25 = (5) × (5)

(c) 36 = (6) × (6), or
36 = (2) × (2) × (3) × (3)

(d) 32 = (2) × (4) × (4), or
32 = (2) × (2) × (2) × (2) × (2)

(e) 20 = (2) × (2) × (5)

(f) 50 = (2) × (5) × (5)

(g) 28 = (2) × (2) × (7)

(h) 90 = (3) × (3) × (10)

(i) 75 = (3) × (5) × (5)

(j) 100 = (10) × (10), or
100 = (2) × (2) × (5) × (5)

(k) 72 = (2) × (2) × (2) × (3) × (3) (there are others)

(l) 144 (12) × (12) (there are others)

(m) 1000 (10) × (10) × (10) (there are others)

(n) 125 (5) × (5) × (5)

Using Exponents to Write Numerals

There is a short way to write product expressions which show repeated factors. This short way uses a numeral to tell the number of times a factor is repeated. Here are some examples.

(a) $5 \times 5 \times 5$ is shortened to 5^3.

(b) 6×6 is shortened to 6^2.

(c) $2 \times 2 \times 2 \times 2 \times 2$ is shortened to 2^5.

What we have is a new way to name numbers. The new symbols like 5^3, 6^2, and 2^5 are made up of two numerals. The upper numeral is called the exponent and the lower one is called the base.

(a) 5^3 is read "five to the third power".

(b) 6^2 is read "six to the second power".

(c) 2^5 is read "two to the fifth power".

The new names are called exponent forms.

(a) 5^3 is the exponent form of the expression $5 \times 5 \times 5$.

$$125 = 5 \times 5 \times 5 = 5^3$$

(decimal) (product expression) (exponent form)

(b) 6^2 is the exponent form of the expression 6×6.

$$36 = 6 \times 6 = 6^2$$

(decimal) (product expression) (exponent form)

(c) 2^5 is the exponent form of the expression
$2 \times 2 \times 2 \times 2 \times 2$.

$$\underset{\text{(decimal)}}{32} = \underset{\text{(product expression)}}{2 \times 2 \times 2 \times 2 \times 2} = \underset{\text{(exponent form)}}{2^5}$$

A number which can be expressed in exponent form is called a power of the number named by the base. The number 125 is called the third power of 5. The number 36 is called the second power of 6, and, the number 64 is called the sixth power of 2. The third power of 4 is also 64. This is why a symbol like 5^3 is read

"five to the third power".

Notice that more than one base can be used in expressing some numbers as powers.

64 is the third power of four and also the sixth power of 2.

$$2^6 = 4^3.$$

1. Express each of the following in words. For 7^3, say "seven to the third power."

(a) 3^5 (three to the fifth power) (e) 10^4 (ten to the fourth power)

(b) 4^2 (four to the second power) (f) 5^3 (five to the third power)

(c) 9^3 (nine to the third power) (g) 12^2 (twelve to the second power)

(d) 7^2 (seven to the second power) (h) 15^3 (fifteen to the third power)

2. Sometimes numbers may be written in several exponent forms: In what different exponent forms is 16 written in Example (a) in the box below? (4^2 and 2^4)

3. In what different exponent forms is 100 written in Example (e)? (10^2) and ($2^2 \times 5^2$)

4. Tell the exponent forms to be used in the blanks in the box.
(see box below)

(a) $16 = 4 \times 4 = 4^2$
$= 2 \times 2 \times 2 \times 2 = 2^4$

(b) $36 = 6 \times 6 = 6^2$
$= 2 \times 2 \times 3 \times 3 = 2^2 \times \underline{(3^2)}$

(c) $28 = 7 \times 2 \times 2 = 7 \times \underline{(2^2)}$

(d) $81 = 9 \times 9 = 9^2$
$= 3 \times 3 \times 3 \times 3 = \underline{(3^4)}$

(e) $100 = 10 \times 10 = 10^2$
$= 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$

(f) $144 = 12 \times 12 = 12^2$
$= 2 \times 2 \times 2 \times 2 \times 3 \times 3 = \underline{(2^4)} \times \underline{(3^2)}$

5. Complete the following sentences:

(a) The numeral 6^3 has exponent (3) and base (6).

(b) The numeral 6^3 is the exponent form of the product expression (6×6×6).

(c) The numeral 6^3 is read (six to the third power).

(d) The number 216 is the (third) power of 6.

(e) In the expression 3×10^4 a (product expression) has been written in exponent form.

(f) If n is a counting number then the number 4^n has (4) as a factor.

(g) The numeral (4^3) has exponent 3 and base 4.

(h) The number 81 can be written in exponent form with base 3 and exponent (4).

(i) The third power of four has decimal numeral (64).

Exercise Set 1

1. Copy and write the product expression for each of the following exponent forms:

Example: $7^3 = 7 \times 7 \times 7$.

(a) 3^4 (c) 6^3 (e) 42^3 (g) 19^4

(b) 5^2 (d) 2^5 (f) 25^2

2. Write the exponent form for each of the following product expressions:

Example: $21 \times 21 \times 21 = 21^3$

(a) $8 \times 8 \times 8 \times 8$ (e) $2 \times 2 \times 2 \times 2 \times 2 \times 2$

(b) $11 \times 11 \times 11$ (f) $30 \times 30 \times 30$

(c) $3 \times 3 \times 3 \times 3 \times 3 \times 3$ (g) $10 \times 10 \times 10 \times 10$

(d) 17×17 (h) $12 \times 12 \times 12 \times 12 \times 12$

3. Express each number below as the product of a repeated factor. Then express it in exponent form. (Hint! If you have trouble finding a repeated factor, express the number as a product of primes.)

Example: $125 = 5 \times 5 \times 5 = 5^3$

(a) 81 (e) 144 (i) 5 to the third power

(b) $6^2 \times 6^2$ (f) 64 (j) 8 to the second power

(c) 32 (g) 625 (k) 10 to the fourth power

(d) 343 (h) 216 (l) 2 to the fifth power

4. Write each of the following product expressions in exponent form as a power of four.

   Example: $4 \times 4 \times 4 = 4^3$

   (a) 4×4
   (b) $4 \times 4 \times 4 \times 4$
   (c) $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$
   (d) 16×64
   (e) $4^2 \times 4^3$
   (f) $4^3 \times 2^3 \times 2$

5. Write the decimal numeral for each of the following:

   Example: $6^3 = 6 \times 6 \times 6 = 36 \times 6 = 216$

   (a) 5^4
   (b) 17^2
   (c) 4^3
   (d) 9^2
   (e) 10^4
   (f) 26^2
   (g) $5^2 \times 2^3$
   (h) $3^2 \times 8^2$

Answers for Exercise Set 1

1. (a) $3^4 = 3 \times 3 \times 3 \times 3$
(b) $5^2 = 5 \times 5$
(c) $6^3 = 6 \times 6 \times 6$
(d) $2^5 = 2 \times 2 \times 2 \times 2 \times 2$
(e) $42^3 = 42 \times 42 \times 42$
(f) $25^2 = 25 \times 25$
(g) $19^4 = 19 \times 19 \times 19 \times 19$

2. (a) 8^4
(b) 11^3
(c) 3^6
(d) 17^2
(e) 2^6
(f) 30^3
(g) 104
(h) 12^5

There are other correct answers for some of these.

3. (a) $9 \times 9 = 9^2$
or $3 \times 3 \times 3 \times 3 = 3^4$
(b) $6 \times 6 \times 6 \times 6 = 6^4$
(c) $2 \times 2 \times 2 \times 2 \times 2 = 2^5$
(d) $343 = 7 \times 7 \times 7 = 7^3$
(e) $12 \times 12 = 12^2$
(f) $8 \times 8 = 8^2$
(g) $25 \times 25 = 25^2$
or $5 \times 5 \times 5 \times 5 = 5^4$
(h) $6 \times 6 \times 6 = 6^3$
(i) $5 \times 5 \times 5 = 5^3$
(j) $8 \times 8 = 8^2$
or $4 \times 4 \times 4 = 4^3$ or 2^6
(k) $10 \times 10 \times 10 \times 10 = 10^4$
(l) $2 \times 2 \times 2 \times 2 \times 2 = 2^5$

4. (a) 4^2
(b) 4^4
(c) 4^7
(d) 4^5
(e) 4^5
(f) 4^8

5. (a) 625
(b) 289
(b) 64
(d) 81
(e) 10,000
(f) 676
(g) 200
(h) 576

POWERS OF TEN

Objective: To help pupils learn how to write decimal numerals in exponent form.

Teaching Procedures:

Writing the expanded notation for decimal numerals in exponent form is especially important. Sentences like:

$$4{,}563 = (4 \times 10^3) + (5 \times 10^2) + (6 \times 10) + (3 \times 1)$$

should help to reinforce the place-value idea.

The development of this section is on P10 and P11. The generalization discussed in Exercises 4-7, page P10 is especially important. The teacher should help every pupil understand this concept.

Exercise Set 2 is practice for this section. Exercise Set 3 is a review of the first two sections and applications of the ideas introduced.

POWERS OF TEN

Table I

A Decimal Numeral	B Product Expression with Repeated Factors	C Exponent Form	D Powers of Ten
(a) 10	None	10^1	First
(b) 100	10 × 10	10^2	(second)
(c) 1,000	10 × 10 × 10	(10^3)	Third
(d) 10,000	(10 x 10 x 10 x 10)	10^4	(fourth)
(e) (100,000)	10 × 10 × 10 × 10 × 10	(10^5)	Fifth
(f) 1,000,000	(10 x 10 x 10 x 10 x 10 x 10)	10^6	(sixth)
(g) (10,000,000)	10 × 10 × 10 × 10 × 10 × 10 × 10	(10^7)	(seventh)
(h) 100,000,000	(10 x 10 x 10 x 10 x 10 x 10 x 10 x 10)	(10^8)	Eighth

1. Read the numerals in Column A above and supply those that are missing.
2. In Column A each number is how many times as large as the one named above it? (10 times)
3. Tell what is missing in Columns A, B, and D.
4. Compare the number of zeros in each numeral in Column A with the exponent of 10 in Column C in the same row. What is true in each comparison? (They are the same.)
5. Do you see that 1 followed by six zeros can be expressed as 10 to the sixth power? (yes) It is written 10^6.
6. To write the decimal numeral for 10^7, we write 1 followed by how many zeros? (seven)
7. Express each of the following as a power of ten.

   (a) 100,000 (10^5) (b) 100,000,000 (10^8) (c) 1,000,000,000 (10^9)

Table II

A Decimal Numeral	B Product Expressions	C Exponent Form of B
400	4 × 100 = 4 × (10 × 10)	4×10^2
6,000	6 × 1,000= 6 × (10 × 10 × 10)	6×10^3
500	5 x 100 = (5 x 10 x 10)	5×10^2
90,000	9 × 10,000 =(9 x 10 x 10 x 10 x 10)	(9×10^4)
(7,000)	7 × 1,000 = 7 × (10 × 10 × 10)	(7×10^3)
300,000	3 × 100,000=(3 x 10 x 10 x 10 x 10 x 10)	(3×10^5)
80	8 × 10	8×10^1
(2,000,000)	(2 x 1,000,000 = 2 x 10 x 10 x 10 x 10 x 10 x 10)	2×10^6
(30)	(3 x 10)	3×10^1
27,000	27 × 1,000 =(27 x 10 x 10 x 10)	27×10^3
15,000,000	15 × 1,000,000 =(15 x 10 x 10 x 10 x 10 x 10 x 10)	15×10^6

1. In the table above, what product expressions are given for 400? (4 x 100 and 4 x 10 x 10)
   How is 400 expressed in exponent form? (4×10^2)
2. How is 6,000 expressed in exponent form? (6×10^3)
3. Supply the numerals which are missing in the above table.
4. On Page 1, the weight of the earth was given as about 13,000,000,000,000,000,000,000,000 pounds. Express the weight in the form used in Column C in the table. (13×10^{24})
5. Did you ever hear the name "googol" used for a number? Googol is the name given to a number written as "1" followed by one hundred zeros. Express this number as a power of ten. (10^{100})

Exercise Set 2

1. Write each of the following in exponent form as a power of ten.

Example: $1{,}000 = 10^3$

(a) 10,000 (10^4) (d) 10 (10^1)

(b) 100 (10^2) (e) 10,000,000,000 (10^{10})

(c) 100,000 (10^5) (f) 1,000,000 (10^6)

2. Write each of the following as a power of 10.

Example: $10 \times 10 = 10^2$

(a) $10 \times 10 \times 10 \times 10$ (10^4)

(b) $10 \times 10 \times 10 \times 10 \times 10 \times 10$ (10^6)

(c) $10 \times 10 \times 10$ (10^3)

(d) $100 \times 100 \times 100$ (10^6)

(e) $10 \times 1{,}000$ (10^4)

(f) $1{,}000 \times 1{,}000 \times 1{,}000$ (10^9)

3. Find the decimal numeral for each of the following.

Example: $6 \times 10^3 = 6{,}000$

(a) 7×10^4 (70,000) (c) 9×10^6 (9,000,000) (e) $(3 \times 2) \times 10^2$ (600)

(b) $10^3 \times 2$ (2000) (d) $10^5 \times 8$ (800000) (f) $(2 \times 5) \times 10^4$ (100,000)

4. Write each of the following in the kind of exponent form shown in exercise 3.

Example: $5{,}000 = 5 \times 10^3$

(a) 60,000 (6×10^4) (d) 8,000,000 (8×10^6)

(b) 200 (2×10^2) (e) 90 (9×10^1)

(c) 700,000 (7×10^5) (f) 300,000,000 (3×10^8)

Exercise Set 3

1. Which of the following is the largest number? (3^4) Which is the smallest number? (3×4) Explain your answer. ($3 \times 3 \times 3 \times 3 = 81$)
($3 \times 4 = 12$)

(a) 3×4 (b) 4^3 (c) 3^4

(d) 43 (e) 34

2. 2^6 is a number how much larger than 6^2? (28)

3. The number 2^8 is how many times as large as the number 8^2? (4 times as large)

4. Suppose you are offered a job which would take you 5 working days to complete. The employer offers you 7¢ the first day. Each day after, for four days, your daily wages will be multiplied by 7.

(a) Make a table like the one below to show the amount you would earn each day. Show also your daily earnings written as a power of 7.

Day on job	Earnings each day	Earnings written as a power of 7
First	7¢	7^1
Second	49¢	7^2
Third	$3.43	7^3
Fourth	$24.01	7^4
Fifth	$168.07	7^5

(b) What will be your total earnings for the week? ($196.07)

5. Find the decimal numeral for each of the following:

(a) 15^2 (225) (c) 2×10 (20)

(b) 3×4^1 (12) (d) $2^3 \times 5^2$ (200)

6. Make the necessary computations. Then mark each of the following mathematical sentences true or false.

(a) $10^2 \not< 2 \times 10$ (T)

(b) $30^2 < 3 \times 10^2$ (F)

(c) $15^2 > 10 + 5^2$ (T)

(d) $3^4 - 2^4 = 50 + 15$ (T)

(e) $11^2 \not< 13^2 - 2^2$ (T)

(f) $6^2 \times 2^2 = 12^2$ (T)

(g) $10^2 - 9^2 = 100 - 90$ (F)

(h) $5^3 + 3^3 < 8^3$ (T)

(i) $150 - 12^2 < 10$ (T)

(j) $9^3 - 700 = 29$ (T)

(k) $8^3 - 80 = 3$ (F)

EXPANDED NOTATION

Objective: To help pupils extend their understanding of the decimal system of numeration by writing numerals in expanded notation using exponents.

Vocabulary: Decimal system, digit, place value, expanded notation

Teaching Procedures:

Most pupils have studied principles of numeration. Here they review them using exponents. An understanding of these properties is one of the very important objectives of the mathematics program.

EXPANDED NOTATION

The system we use for naming numbers is the decimal system. In our system we group by tens. The word decimal comes from the Latin word "decem" which means "ten."

Just as our written language uses an alphabet of 26 symbols, the decimal system uses an "alphabet" of ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These are used as numerals for digits. Digits are whole numbers less than ten.

In our written language the alphabet symbols are used to form words which are used as names. In the decimal system the ten symbols for digits are used to form "words" which name larger whole numbers. These "words" are numerals made up of two, three, four, or more digit numerals.

To understand the decimal numeral system we learn how to find the meaning of "words" like 23, 8.6, .04. Let us review the way we think of decimals for whole numbers.

1. In the numeral 5555, each numeral 5 represents a different value. The place in which a 5 is written tells whether it represents 5 ones, 5 tens, 5 hundreds or 5 thousands. The meaning of each numeral is shown by the diagram in box A. Read the names of the places shown in box A.

A

5 5 5 5

5	ones	or (5×1)	
5	tens	or (5×10)	or (5×10^1)
5	hundreds	or $(5 \times 10 \times 10)$	or (5×10^2)
5	thousands	or $(5 \times 10 \times 10 \times 10)$	or (5×10^3)

2. As we go from right to left in the numeral 5555, the value represented by each 5 is how many times the value of the 5 before it? (10) As we go from left to right the value represented by each 5 is one-tenth the value of the 5 before it.

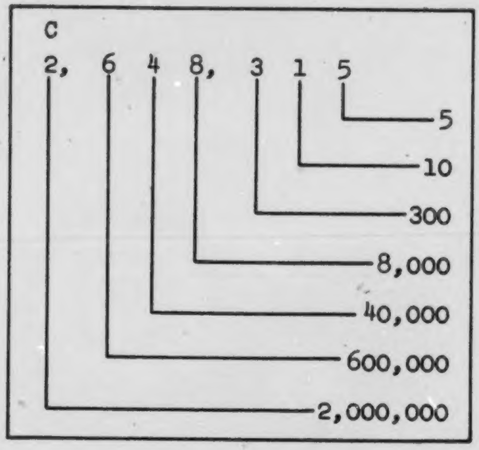
3. The diagram in box A shows that place values are powers of ten. In the decimal system, we group not just by tens, but by powers of ten. What powers of ten are shown in box A? (first, second and third powers)

B

$$5555 = (5 \times 10 \times 10 \times 10) + (5 \times 10 \times 10) + (5 \times 10) + (5 \times 1)$$

$$= (5 \times 10^3) + (5 \times 10^2) + (5 \times 10^1) + (5 \times 1)$$

Box B shows the numeral 5555 written in expanded form or in expanded notation. The last line shows the exponent form of this expanded notation.

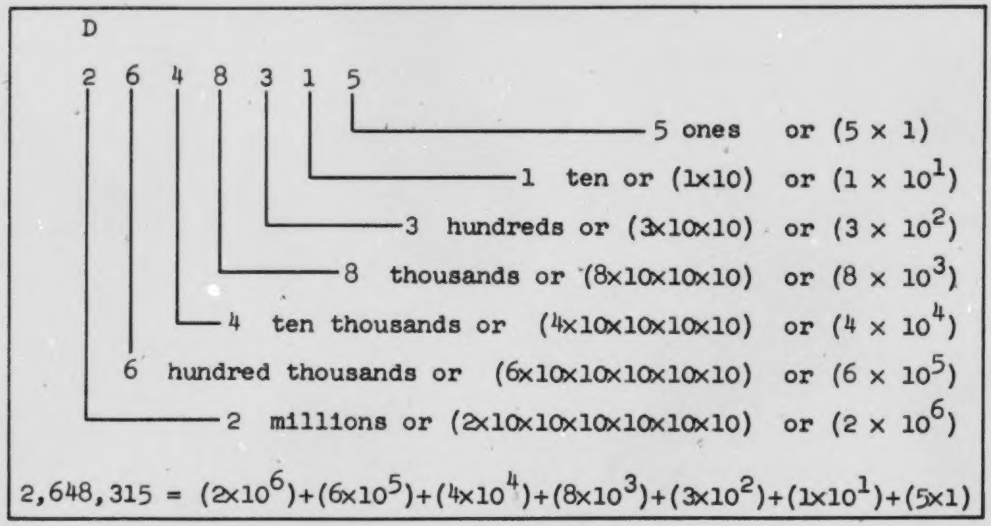


The numeral 2,648,315 is read "two million, six hundred forty eight thousand, three hundred fifteen."

Each group of three-place numerals is separated by a comma to make reading easier. We do not use the word "and" between each group because "and" is reserved for use in reading the decimal point in numerals such as 123.85.

4. Study the diagram in box C and tell the value represented by each digit in 2,648,315. (See box C)

5. Now study the diagram in box D and tell the value represented by each digit numeral in 2,648,315. Give the value in repeated factor form and in exponent form.



D

2 6 4 8 3 1 5

5 ones or (5×1)

1 ten or (1×10) or (1×10^1)

3 hundreds or $(3\times10\times10)$ or (3×10^2)

8 thousands or $(8\times10\times10\times10)$ or (8×10^3)

4 ten thousands or $(4\times10\times10\times10\times10)$ or (4×10^4)

6 hundred thousands or $(6\times10\times10\times10\times10\times10)$ or (6×10^5)

2 millions or $(2\times10\times10\times10\times10\times10\times10)$ or (2×10^6)

$$2{,}648{,}315 = (2\times10^6)+(6\times10^5)+(4\times10^4)+(8\times10^3)+(3\times10^2)+(1\times10^1)+(5\times1)$$

6. Read each of the following numerals:

(a)	317	(c)	1,306	(e)	10,010	(g)	606,606
(b)	98	(d)	26,840	(f)	545,845	(h)	32,976,418

7. Write a decimal numeral for each of these:

(a) nine hundred three (903)

(b) thirty thousand three hundred thirty (30,330)

(c) eight thousand eight (8,008)

(d) four hundred forty five thousand four hundred forty five (445,445)

8. Express each of the following numerals in expanded notation. Give both the repeated factor form and also the exponent form.

(a) 783 $(7\times10\times10)+(8\times10)+(3\times1)$ or $(7\times10^2)+(8\times10)+(3\times1)$

(b) 3,075 $(3\times10\times10\times10)+(0\times10\times10)+(7\times10)+(5\times1)$ or $(3\times10^3)+(0\times10^2)+(7\times10)+(5\times1)$

(c) 81,040 $(8\times10^4)+(1\times10^3)+(0\times10^2)+(4\times10)+(0\times1)$ or $(8\times10\times10\times10\times10)+(1\times10\times10\times10)+(0\times10\times10)+(4\times10)+(0\times1)$

(d) 200,456 $(2\times10\times10\times10\times10\times10)+(0\times10\times10\times10\times10)+(0\times10\times10\times10)+(4\times10\times10)+(5\times10)+(6\times1)$ or $(2\times10^5)+(0\times10^4)+(0\times10^3)+(4\times10^2)+(5\times10)+(6\times1)$

(e) 73,800 $(7\times10\times10\times10\times10)+(3\times10\times10\times10)+(8\times10\times10)+(0\times10)+(0\times1)$ or $(7\times10^4)+(3\times10^3)+(8\times10^2)+(0\times10)+(0\times1)$

(f) 5,247,600 $(5\times10^6)+(2\times10^5)+(4\times10^4)+(7\times10^3)+(6\times10^2)+(0\times10)+(0\times1)$ or $(5\times10\times10\times10\times10\times10\times10)+(2\times10\times10\times10\times10\times10)+(4\times10\times10\times10\times10)+(7\times10\times10\times10)+(6\times10\times10)+(0\times10)+(0\times1)$

Summary:

1. Grouping in the decimal system is by tens and powers of ten.
2. The decimal system has ten special symbols for the ten digits of the system.
3. In the decimal system the place values are powers of ten arranged in increasing order from right to left.
4. The place names from right to left are units (ones), tens, hundreds, thousands, ten thousands, hundred thousands, millions, and so on.

Answers for Exercise Set 4

1. (a) $(6 \times 100) + (7 \times 10) + (5 \times 1)$

$(6 \times 10 \times 10) + (7 \times 10) + (5 \times 1)$

$(6 \times 10^2) + (7 \times 10) + (5 \times 1)$

(b) $(8 \times 1000) + (0 \times 100) + (4 \times 10) + (2 \times 1)$

$(8 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (4 \times 10) + (2 \times 1)$

$(8 \times 10^3) + (0 \times 10^2) + (4 \times 10) + (2 \times 1)$

(c) $(5 \times 1000) + (0 \times 100) + (6 \times 10) + (8 \times 1)$

$(5 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (6 \times 10) + (8 \times 1)$

$(5 \times 10^3) + (0 \times 10^2) + (6 \times 10^1) + (8 \times 1)$

(d) $(2 \times 10{,}000) + (6 \times 1000) + (4 \times 100) + (0 \times 10) + (5 \times 1)$

$(2 \times 10 \times 10 \times 10 \times 10) + (6 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (0 \times 10) + (5 \times 1)$

$(2 \times 10^4) + (6 \times 10^3) + (4 \times 10^2) + (0 \times 10) + (5 \times 1)$

(e)
$(1 \times 100{,}000) + (3 \times 10{,}000) + (7 \times 1000) + (6 \times 100) + (0 \times 10) + (0 \times 1)$

$(1 \times 10 \times 10 \times 10 \times 10 \times 10) + (3 \times 10 \times 10 \times 10 \times 10) + (7 \times 10 \times 10 \times 10) + (6 \times 10 \times 10)$
$+ (0 \times 10) + (0 \times 1)$

$(1 \times 10^5) + (3 \times 10^4) + (7 \times 10^3) + (6 \times 10^2) + (0 \times 10) + (0 \times 1)$

(f) $(2 \times 1{,}000{,}000) + (9 \times 100{,}000) + (8 \times 10{,}000) + (7 \times 1000)$
$+ (6 \times 100) + (5 \times 10) + (4 \times 1)$

$(2 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10) + (9 \times 10 \times 10 \times 10 \times 10 \times 10) + (8 \times 10 \times 10 \times 10 \times 10)$
$+ (7 \times 10 \times 10 \times 10) + (6 \times 10 \times 10) + (5 \times 10) + (4 \times 1)$

$(2 \times 10^6) + (9 \times 10^5) + (8 \times 10^4) + (7 \times 10^3) + (6 \times 10^2)$
$+ (5 \times 10^1) \times (4 \times 1)$

2. (a) 365 (b) 478 (c) 8067 (d) 23,540

(e) 9,352 (f) 60,408 (g) 73,000

3. The answers will vary. Here are a few:
billions, trillions, quadrillions, quintillions, sextillions, septillions, octillions etc.

Exercise Set 4

The numeral, 234, has been written in expanded notation in three ways in the box at the right.

$$\begin{aligned} 234 &= (2\times100) + (3\times10) + (4\times1) \\ &= (2\times10\times10) + (3\times10) + (4\times1) \\ &= (2\times10^2) + (3\times10^1) + (4\times1) \end{aligned}$$

1. For each of the following numerals, write the expanded notation in the three ways shown in the example above.

   (a) 675 (d) 26,405

   (b) 8042 (e) 137,600

   (c) 5,168 (f) 2,987,654

2. Write the decimal numeral which is expressed in expanded notation below.

   Examples: $(6 \times 10^3) + (4 \times 10^2) + (2 \times 10^1) + (1 \times 1) = 6421$

   $(7 \times 10 \times 10) + (0 \times 10) + (8 \times 1) = 708$

   (a) $(3 \times 10^2) + (6 \times 10) + (5 \times 1)$

   (b) $(4 \times 10^2) + (7 \times 10) + (8 \times 1)$

   (c) $(3 \times 10^3) + (0 \times 10^2) + (6 \times 10) + (7 \times 1)$

   (d) $(2 \times 10^4) + (3 \times 10^3) + (5 \times 10^2) + (4 \times 10) + (0 \times 1)$

   (e) $(9 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$

   (f) $(6 \times 10 \times 10 \times 10 \times 10) + (0 \times 10^3) + (4 \times 10 \times 10) + (0 \times 10) + (8 \times 1)$

   (g) $(7 \times 10^4) + (3 \times 10^3) + (0 \times 10^2) + (0 \times 10^1) + (0 \times 1)$

3. Find the names for as many groups beyond the million group as you can.

Exercise Set 5

1. Name the largest and the smallest numbers which have exactly four decimal numerals, not using any zero. (9999) (1111)

2. Use the numerals 4, 5, 6, 7, and 8 to name eight different numbers with five-place numerals. Write the numerals in a column in order of the size of the numbers from smallest to largest.
(answers will vary)

3. Show that 2^5 and 5^2 do not name the same number.
$(2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32, \quad 5^2 = 5 \times 5 = 25)$

4. Do 2^4 and 4^2 name the same number? (yes)

5. Make the necessary computations, then mark each of the following mathematical sentences true or false.

(a) $4^3 = 8^2$ (T)
(b) $4 \times 10 = 400$ (F)
(c) $10 \times 12 < 12^2$ (T)
(d) $10^3 > 5 \times 100$ (T)
(e) $7 \times 2 \times 2 \times 2 > 7 \times 2^2$ (T)
(f) $2 \times 6^2 = 3^2 \times 2^4$ (F)
(g) $6^2 \neq 2^6$ (T)
(h) $9^2 = 3^4$ (T)

6. Copy the following and make each one into a true mathematical sentence. Do this by writing one of the symbols in the box in each blank.

Symbols
>
<
=

(a) 3×10^4 $\underline{(=)}$ $6 \times 5 \times 10^3$
(b) 7×10^2 $\underline{(\angle)}$ $2^3 \times 100$
(c) $2^3 \times 3^2$ $\underline{(>)}$ 4×4^2
(d) $(6 \times 10^2) + 5^2$ $\underline{(=)}$ 5^4
(e) $10^2 \times 7^2$ $\underline{(\angle)}$ 3×12^3
(f) 3×50^2 $\underline{(\angle)}$ $9^2 \times 10^2$

PRODUCTS EXPRESSED IN EXPONENT FORM

Objective: To help pupils understand and use the rule for computing products using exponent forms.

Teaching Procedures:

> Pupils can discover the rule for multiplying numbers expressed in exponent form. Below is a suggested introduction. It is written as though the teacher were speaking to the class. Possible pupil answers are enclosed in parentheses. Pupils should have their books closed.

You know how to name the product of two numbers using decimal numerals. Guess what this product is. Could you guess the answer to this question, "Find an exponent form for n if $3^6 \times 3^{10} = n$?" (Urge pupils to guess. Possible answers might be 3^{60}, 9^{60}, 9^{16} or even the correct answer 3^{16}) Give me some reasons why you think your answer is correct.

Let us put this exercise aside and try some easier ones. Maybe we can discover how to name the product.

> The teacher may now find products such as $2^3 \times 2^1$, $3^2 \times 3^4$, $10^1 \times 10^2$ using the procedures illustrated in the box on page P21. Then pages P21 and P22 should be studied carefully. Then examples like $2^8 \times 2^{10} = 2^{18}$ can readily be done.

FINDING PRODUCTS USING EXPONENT FORMS

1. Can you think of a way to find an exponent form for these product expressions?

   (a) $10^2 \times 10^2 = ?$ (10^4) (c) $10^5 \times 10^3 = ?$ (10^8)

   (b) $2^3 \times 2^2 = ?$ (2^5) (d) $7^3 \times 7^1 = ?$ (7^4)

We could solve the problems above by changing the exponent forms to decimal numerals, then multiplying in the usual way and then changing back to exponent form.

$$2^3 \times 2^2 = 8 \times 4$$
$$= 32.$$

Then since

$$32 = 8 \times 4$$
$$= (2 \times 2 \times 2) \times (2 \times 2)$$
$$= 2^5$$
$$2^3 \times 2^2 = 2^5$$

It will be much quicker if we can learn to name the product in exponent form without changing to decimal numerals and back. Finding a way to do this and learning to use it is our purpose in this section.

2. In the box below are examples of the multiplication of numbers expressed in exponent forms with the same base. Study these examples. Can you discover how the exponent of the numeral for the product is obtained? The questions below the box may help.

(a) $2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2^7$
(b) $3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) = 3^5$
(c) $10^2 \times 10^4 = (10 \times 10) \times (10 \times 10 \times 10 \times 10) = 10^6$

3. Example (a) shows exponent forms for two numbers, 2^3 and 2^4, and for their product, 2^7. What is true about the base of all three forms? How can the exponent for the product be found from the exponents for the factors?

4. In example (b) the base in each exponent form is 3. The three exponents shown are 2, 3, and 5. What is the exponent in the product? (5) What addition fact connects these exponents? $(2+3=5)$

5. In example (c) the exponent 6 was found by counting. What operation can be used instead of counting 2 and then 4?

6. For the following examples, change each exponent form to repeated factor form. Are all the products correct? (no) (d is not correct)

   (a) $5^3 \times 5^1 = 5^4$

   (b) $7^2 \times 7^2 = 7^4$

   (c) $6^1 \times 6^1 \times 6^1 = 6^3$

   (d) $10^2 \times 10^2 \times 10^2 = 10^5$

   (e) $8^3 \times 8^2 = 8^5$

   (f) $4^2 \times 4^3 = 4^5$

7. For the following examples, write the number as a power without changing to repeated factor form.

   Example: $2^4 \times 2^3 = 2^{(4+3)} = 2^7$

   (a) $6^2 \times 6^1$ $(6^{(2+1)} = 6^3)$

   (b) $3^3 \times 3^1 \times 3^2$ $(3^{(3+1+2)} = 3^6)$

   (c) $5^2 \times 5^2 \times 5^2$ $(5^{(2+2+2)} = 5^6)$

   (d) $10^3 \times 10^2 \times 10^2$ $(10^{(3+2+2)} = 10^7)$

   (e) $25^2 \times 25^2$ $(25^{(2+2)} = 25^4)$

   (f) $100^2 \times 100^2$ $(100^{(2+2)} = 100^4)$

8. Copy and complete each of the following. Use only exponent forms.

(a) $2^1 \times 2^3 = \underline{(2^4)}$

(b) $18^1 \times 18^4 = \underline{(18^5)}$

(c) $\underline{(9^5)} \times 9^1 = 9^6$

(d) $a^1 \times a^7 = \underline{(a^8)}$

(e) $6 \times 6^3 = \underline{(6^4)}$

(f) $2^4 \times 2 = \underline{(2^5)}$

(g) $3 \times \underline{(3^2)} = 3^3$

(h) $a^5 \times a = \underline{(a^6)}$

(i) $5^4 \times \underline{(5^2)} = 5^6$

(j) $27^3 \times 27^6 = \underline{(27^9)}$

(k) $19^5 \times \underline{(19^3)} = 19^8$

FINDING A COMMON BASE

1. Write each of the following as a power of 2.

   Example: $16 = 4 \times 4 = (2 \times 2) \times (2 \times 2) = 2^4$, or

   $16 = 2 \times 8 = 2 \times 2 \times 4 = 2 \times 2 \times 2 \times 2 = 2^4$

   (a) 64 (2^6) (b) 32 (2^5) (c) 128 (2^7)

2. Write each of the following as a power of 5.

   (a) 25 (5^2) (b) 625 (5^4) (c) 125 (5^3)

3. Write each of the following as a power of 10.

   (a) 100 (10^2) (b) 10,000 (10^4) (c) 1,000 (10^3)

One way to find an exponent form for 125×25 is first to change the decimal numerals 125 and 25 to exponent forms with the same base. Here are two examples of this method.

$$\text{(a)}\quad 125 \times 25 = 5^3 \times 5^2 = 5^5$$

$$\text{(b)}\quad 49 \times 343 = 7^2 \times 7^3 = 7^5$$

This method can be used only if the factors in the product expression are powers of the same number.

4. Express as powers of the same number and multiply using exponent forms.

   (a) $16 \times 4^2 = (4^2 \times 4^2 = 4^4)$

   (b) $7^3 \times 49 = (7^3 \times 7^2 = 7^5)$

   (c) $5^3 \times 25 = (5^3 \times 5^2 = 5^5)$

   (d) $81 \times 3^3 = (3^4 \times 3^3 = 3^7)$

   (e) $64 \times 32 = (2^6 \times 2^5 = 2^{11})$

   (f) $3 \times 9^2 = (3 \times 3^4 = 3^5)$

Exercise Set 6

1. Write each of these numbers as a power.

(a) $5^2 \times 5^8 = (5^{10})$

(b) $3^4 \times 3^2 = (3^6)$

(c) $9 \times 9^2 = (9^3)$

(d) $10^5 \times 10^5 = (10^{10})$

(e) $7^2 \times 7 = (7^3)$

(f) $6^2 \times 6^8 = (6^{10})$

(g) $8^3 \times 8^4 = (8^7)$

2. Change the numerals to exponent forms with the same base and multiply.

Example: $81 \times 27 = 3^4 \times 3^3 = 3^7$.

(a) $9 \times 3 = (3^2 \times 3^1 = 3^3)$

(b) $8 \times 2 = (2^3 \times 2^1 = 2^4)$

(c) $25 \times 25 = (25^2)$ or $(5^2 \times 5^2 = 5^4)$

(d) $81 \times 9 = (9^2 \times 9^1 = 9^3)$ or
$(3^4 \times 3^2 = 3^6)$

(e) $32 \times 4 = (2^5 \times 2^2 = 2^7)$

(f) $64 \times 16 = (4^3 \times 4^2 = 4^5)$ or $(2^6 \times 2^4 = 2^{10})$

(g) $100 \times 10{,}000 = (10^2 \times 10^4 = 10^6)$

(h) $10{,}000 \times 10{,}000 = (10^4 \times 10^4 = 10^8)$

3. Think of letters of our alphabet as names of counting numbers. Express each of the following as a power.

Example: $a^2 \times a^3 = a^{(2+3)} = a^5$

(a) $b^2 \times b^3 = (b^5 = b^{2+3})$

(b) $y^3 \times y^2 = (y^{(3+2)} = y^5)$

(c) $4^2 \times 4^3 = (4^{(2+3)} = 4^5)$

(d) $n^3 \times n^4 = (n^{(3+4)} = n^7)$

(e) $3^4 \times 3^7 = (3^{(4+7)} = 3^{11})$

(f) $m^5 \times m^2 = (m^{(5+2)} = m^7)$

4. Use exponent forms to shorten the multiplication process as shown in the example.

Example: $300 \times 4{,}000 = (3 \times 10^2) \times (4 \times 10^3)$

$= (3 \times 4) \times 10^2 \times 10^3 = 12 \times 10^5$

$= 1{,}200{,}000$

(a) $50 \times 700 =$ $(5 \times 10) \times (7 \times 10^2)$
$= (5 \times 7) \times (10 \times 10^2)$
$= 35 \times 10^3 = 35{,}000$

(b) $400 \times 400 =$ $(4 \times 10^2) \times (4 \times 10^2)$
$= (4 \times 4) \times (10^2 \times 10^2)$
$= 16 \times 10^4 = 160{,}000$

(c) $500 \times 60 =$ $(5 \times 10^2) \times (6 \times 10) = (5 \times 6) \times (10^2 \times 10) =$
$30 \times 10^3 = 30{,}000$

(d) $1{,}600 \times 500 =$ $(16 \times 10^2) \times (5 \times 10^2)$
$= (16 \times 5) \times (10^2 \times 10^2)$
$= 80 \times 10^4 = 800{,}000$

QUOTIENTS EXPRESSED IN EXPONENT FORM

Objective: To help pupils understand and use the rule for finding the quotient when the dividend and divisor are expressed in exponent form.

Teaching Procedure:

Suggestions for the previous section are also appropriate for this section. Pupils should be able to make more intelligent guesses for quotients to exercises such as $2^6 \div 2^2$ because of their experience with multiplication.

As with other concepts developed in this unit, the ideas are especially important. Skill in using the rule is not expected from most pupils. However, you should try to help all pupils formulate answers to such questions as:

1. Why is $3^6 \div 3^2 = 3^4$?
2. Is $2^6 \div 2^2$ equal to $2^{6 - 2}$ or $2^{6 \div 2}$?
3. Why is $5^4 \div 5^4 = 1$?
4. Which is larger $4^6 \div 4^2$ or $4^6 \div 4^3$? Why?
5. Is $1^5 \div 1^3$ equal to $1^8 \div 1^7$? Why?

QUOTIENTS EXPRESSED IN EXPONENT FORM

Since 216 ÷ 36 names the number 6 as a quotient we will call it a quotient expression.

You have learned how to find the decimal numeral for 216 ÷ 36 by the division process. Suppose 216 were written in exponent form as 6^3 and 36 were written as 6^2. Is there a way to divide as well as multiply using exponent forms? Can we fill in the blank below with an exponent form?

$$6^3 \div 6^2 = \underline{\qquad}$$

Here are examples showing two ways we might answer such a question.

First Way

(a) $4^5 \div 4^2 = (4 \times 4 \times 4 \times 4 \times 4) \div (4 \times 4)$

$= (64 \times 4 \times 4) \div 16 = (256 \times 4) \div 16$

$= 1024 \div 16$

$= 64$

$= 4^3$

(b) $2^7 \div 2^3 = \big((2 \times 2) \times (2 \times 2) \times (2 \times 2) \times 2\big) \div (2 \times 2 \times 2)$

$= (4 \times 4 \times 4 \times 2) \div 8$

$= (64 \times 2) \div 8$

$= 128 \div 8$

$= 16$

$= 2^4$

Second Way

(a) $4^5 \div 4^2 = (4 \times 4 \times 4 \times 4 \times 4) \div 4^2$

$= \big((4 \times 4 \times 4) \times (4 \times 4)\big) \div 4^2 = (4^3 \times 4^2) \div 4^2$

We notice that we must multiply by 4^2 and then divide the result by 4^2. But these operations undo one another, so we do not need to do either.

$(4^3 \times 4^2) \div 4^2 = 4^3$.

$4^5 \div 4^2 = 4^3$.

(b) To express $2^7 \div 2^3$ as a power of 2 we use $2^n \times 2^3 = 2^7$. Since $n + 3 = 7$, $n = 4$ and $2^4 \times 2^3 = 2^7$. So $2^7 \div 2^3 = (2^4 \times 2^3) \div 2^3 = 2^4$.

Since the second way is so much shorter than the first it is the way we *should* use if we can understand it. Perhaps we can see better how it works if we first write division sentences as multiplication sentences:

(a) To find $4^5 \div 4^2$ we think $? \times 4^2 = 4^5$. Now we think the ? can be 4^n and write $4^n \times 4^2 = 4^5$.

Since $n + 2 = 5$, $n = 3$ and $4^3 \times 4^2 = 4^5$. So

$$4^5 \div 4^2 = (4^3 \times 4^2) \div 4^2 = 4^3$$

We now write $4^5 \div 4^2 = 4^3$

(b) If $n = 2^7 \div 2^3$ then

$n \times 2^3 = 2^7$.

But $2^4 \times 2^3 = 2^7$, so

$n = 2^4 = 2^{(7 - 3)}$.

1. Use the ideas in the above examples in explaining how to fill in each blank with an exponent form.

(a) $5^3 \times 5 = \underline{(5^4)}$

(b) $(5^3 \times 5) \div 5 = \underline{(5^3)}$

(c) $5^4 \div 5 = \underline{(5^3)}$

(d) $(3^3 \times 3^2) \div 3^2 = \underline{(3^3)}$

(e) $3^5 \div 3^2 = \underline{(3^3)}$

(f) $10^4 \times 10^2 = \underline{(10^6)}$

(g) $10^4 \div 10^2 = \underline{(10^2)}$

(h) $10^4 \times \underline{(10^2)} = 10^6$

(i) $10^6 \div 10^4 = \underline{(10^2)}$

(j) $10^{(2+4)} \div \underline{(10^4)} = 10^2$

Exercise Set 7

1. Write each quotient expression in exponent form.

(a) $2^5 \div 2^3 = (2^2)$

(b) $4^4 \div 4^1 = (4^3)$

(c) $3^6 \div 3^3 = (3^3)$

(d) $10^3 \div 10^2 = (10^1)$

(e) $9^2 \div 9 = (9^1)$

(f) $7^3 \div 7^2 = (7^1)$

(g) $10^6 \div 10^3 = (10^3)$

(h) $12^5 \div 12^3 = (12^2)$

2. Change the numerals to exponent forms with the same base and divide.

Example: $216 \div 36 = 6^3 \div 6^2 = 6^{(3 - 2)} = 6^1 = 6$

(a) $16 \div 4 = (4^2 \div 4^1 = 4^1)$

(b) $64 \div 2 = (2^6 \div 2^1 = 2^5)$

(c) $243 \div 9 = (3^5 \div 3^2 = 3^3)$

(d) $10{,}000 \div 1{,}000 = (10^4 \div 10^3 = 10^1)$

(e) $81 \div 27 = (3^4 \div 3^3 = 3^1)$

(f) $1{,}000{,}000 \div 100 = (10^6 \div 10^2 = 10^4)$

3. Answer the following in exponent form.

(a) $10^6 \times 10^3 = (10^9)$

(b) $15^3 \div 15^2 = (15^1)$

(c) $2^8 \div 2^4 = (2^4)$

(d) $10 \times 10 = (10^2)$

(e) $16 \div 8 = (2^1)$

(f) $9^2 \times 3^3 = (3^7)$

(g) $4^3 \times 64 = (4^6)$

(h) $16 \div 16 = (1^1$ or $1)$

(i) $5^3 \div 5 = (5^2)$

(j) $1^4 \div 1^2 = (1^2)$

Exercise Set 8

Using Exponent Forms

1. The area of a square region is 5^6 square feet. How long is a side? (5^3 ft.)

2. A rectangular region has sides which are 2^4 inches and 5^4 inches.

   (a) Name the measure of the area of the region in any convenient way, ($2^4 \times 5^4$) or (10^4)

   (b) Write the decimal numeral for the area measure. (10,000)

3. The area of the United States is about 3,600,000 square miles. If our country were a rectangular region with one side 1,000 miles long, how long would the other side be? (3,600 miles)

4. Some very small animals which can be seen only through a microscope increase in number by splitting into two of the same kind. After a certain time each of these divides into two animals and so on. Suppose one kind of such animals divides exactly every 10 minutes.

   (a) How many animals will be produced from a single animal in one hour? (2^6 or 64)

   (b) About how long is required to produce 1,000 animals from 1 animal? (1 hour and 40 minutes)

5. To go into orbit around the earth a satellite must be travelling about 18×10^3 miles an hour. In circling the earth once the satellite goes about 27×10^3 miles. How many times around the earth does the satellite go in 3 hours? (twice)

6. The nearest star is about $3,441 \times 10^{10}$ miles away. The sun is about 93×10^6 miles away.
   (a) Write the decimal numeral for 93×10^6. (93,000,000)
   (b) About how many times as far away as the sun is the nearest star? (37×10^4) or (370,000)
   (c) If the distance to the sun were used as a unit, about what would be the measure of the distance to the nearest star? (37×10^4) or (370,000 units)

7. Light travels about 186,000 miles a second.
   (a) About how many seconds does it take light to travel from the sun to the earth? (500 seconds) How many minutes, to the nearest minute? (8 minutes)
   (b) Use the answers to 7 (a) and 6 (b) to find about how many minutes it takes light to travel from the the nearest star to the earth. ($37 \times 10^4 \times 8$ or 2,960,000 min)
   (c) Find out whether this is longer or shorter than one year. (longer. It is over 4 years There are approximately 525,600 minutes in a year)

Supplementary Exercise Set

1. For each whole number from 50 through 70, write the number as a product of primes, or write "prime" after the number if it is prime. If a prime factor occurs more than once, rewrite the product expression using exponential forms.

   Here is an example: $50 = 2 \times 5 \times 5$
   $= 2 \times 5^2$.

2. Express each of the following numbers as a product of powers of primes as in the example.

   Example: $(2^3 \times 3^2) \times (2 \times 3^3 \times 5) = 2^4 \times 3^5 \times 5$.

   (a) $2^3 \times (2 \times 3^4) = (2^4 \times 3^4)$
   (b) $5^2 \times (3 \times 5 \times 7) = (3 \times 5^3 \times 7)$
   (c) $3^2 \times 48 = (3^3 \times 2^4)$
   (d) $36 \times (2 \times 5^2) = (2^3 \times 3^2 \times 5^2)$
   (e) $36 \times 48 = (2^6 \times 3^3)$
   (f) $10^2 \times 6 \times 7 = (2^3 \times 3 \times 5^2 \times 7)$
   (g) $10^2 \times 2^2 \times 5 = (2^4 \times 5^3)$
   (h) $144 \times 12 = (3^3 \times 2^6)$
   (i) The number of minutes in a day $= (2^5 \times 3^2 \times 5)$

3. Write "yes" if the second number is a factor of the first. Write "no" if it is not. Do not make any long computations.

   (a) $2^2 \times 3 \times 5^3$, 2^2 (yes)
   (b) $2^2 \times 3 \times 5^3$, 3×5^3 (yes)
   (c) $2^2 \times 3 \times 5^3$, 14 (no)
   (d) $2^2 \times 3 \times 5^3$, 6 (yes)
   (e) 6×5^2, 15 (yes)
   (f) 60×60, 25 (yes)
   (g) $2 \times 3^2 \times 5 \times 7^2$, 35 (yes)

answers for 1.

$50 = 2 \times 5^2$
$51 = 3 \times 17$
$52 = 2^2 \times 13$
53 prime
$54 = 2 \times 3^3$
$55 = 5 \times 11$
$56 = 2^3 \times 7$
$57 = 3 \times 19$
$58 = 2 \times 29$
59 prime
$60 = 3 \times 4 \times 5$
61 prime
$62 = 2 \times 31$
$63 = 3^3 \times 7$
$64 = 2^6$
$65 = 5 \times 13$
$66 = 2 \times 3 \times 11$
67 prime
$68 = 2^2 \times 17$
$69 = 3 \times 23$
$70 = 2 \times 5 \times 7$

4. Write T for the following sentences which are true and F for those which are false.

(T) (a) $9 \times 10^3 < 10^4$

(F) (b) $(3 + 4)^2 = 3^2 + 4^2$

(F) (c) $17^5 > 17^6$

(F) (d) $4^2 \times 10^2 > 2,500$

(F) (e) $3^6 \div 3^2 = 3^3$

(T) (f) $10^3 = 2^3 \times 5^3$

(F) (g) $(m \times n)^3 = m^3 \times n^3$

(T) (h) $(4^2)^3 = 4^2 \times 4^2 \times 4^2$

(F) (i) $(4^2)^3 = 4^5$

Chapter 2

MULTIPLICATION OF RATIONAL NUMBERS

PURPOSE OF UNIT

The purpose of this unit is to extend the concept of rational number and operations on rational numbers. Although the primary purpose of the unit is to develop properties and techniques of the multiplication of rational numbers, opportunity is also provided to review the idea of rational number associating it with regions, segments, and sets of objects. Some practice is given in addition and subtraction of rational numbers. Following this review, the specific purposes are:

(1) To develop meaning for the multiplication of rational numbers. This is done by associating the product of two rational numbers (a) with a rectangular region, (b) with segments on the number line, and (c) with reference to a collection of objects.

(2) To develop methods for computing the product of rational numbers named by fractions, decimals, and mixed forms (for example such numerals as $\frac{2}{3}$, 1.5, $2\frac{1}{4}$).

(3) To develop ability to solve problems relating to situations involving multiplication of rational numbers.

Introduction

In the study of mathematics in the elementary school, a child learns to use several sets of numbers. The first of these is the set of counting numbers, 1, 2, 3, 4, ... The second is the set of whole numbers, 0, 1, 2, 3, 4, ... The child also may have learned certain properties of whole numbers.

During the primary and middle grades the idea of "number" is enlarged, so that by the end of the sixth grade the child recognizes each of the following as a name for a number:

$$\frac{4}{3},\ \frac{1}{2},\ 3.6,\ 2\frac{1}{2},\ 8,\ 0,\ \frac{5}{5},\ \frac{6}{2},\ .01$$

In traditional language, we might say that when the child has completed the first six years of school mathematics he knows about "the whole numbers, fractions, decimals, and mixed numbers." This language is primarily numeral language. It obscures the fact that a single number can have names of many kinds. "Fractions, decimals, and mixed numbers" are kinds of number names rather than different kinds of numbers. Whether we make a piece of ribbon $1\frac{1}{2}$ in. long, or 1.5 in. long, or $\frac{3}{2}$ in. long makes no difference--our ribbon is the same whatever our choice of numeral. That is, $1\frac{1}{2}$, 1.5, $\frac{3}{2}$ are all names for the same number. This number is a member of a set of numbers sometimes called the non-negative numbers or the rational numbers of arithmetic. For our purposes here, we shall call them the rational numbers, realizing that they are only a subset of the set of all rational numbers. It also should be realized that within the set of rational numbers is a set which corresponds to the set of whole numbers. For example, 0, 3, 7 are all rational numbers that are also whole numbers. $\frac{3}{4}$, $\frac{7}{4}$, and .2 are rational numbers that are not whole numbers.

First Ideas About Rational Numbers

Children develop early ideas about rational numbers by working with regions--rectangular regions, circular regions, triangular regions, etc. In Figures A, B, and C, rectangular regions have been used. For any type of region we must first identify the unit region. In Figures A, B, and C, the unit region is a square region.

In Figures A and B we see that:

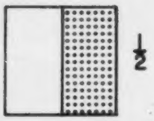
(1) The unit region has been separated into a number of congruent regions.

(2) Some of the regions have been shaded.

(a) Using regions. Let us see how children use regions to develop their first ideas of rational numbers. The child learns in simple cases to associate a number like $\frac{1}{2}$ or $\frac{2}{3}$ with a shaded portion of the figure. (Rational numbers can also be associated with the unshaded portions.)

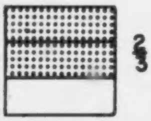
Using two or more congruent regions (Fig. C), he can separate each into the same number of congruent parts and shade some of the parts. Again, he can associate a number with the resulting shaded region.

Fig. A



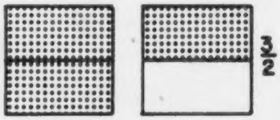
The unit square is separated into 2 congruent regions. 1 is shaded.

Fig. B



The unit square is separated into 3 congruent regions. 2 are shaded.

Fig. C



Each unit square is separated into 2 congruent regions. 3 are shaded. We have $\frac{3}{2}$ of a unit square.

At this point, the child is only at the beginning of his concept of rational numbers. However, let us note what we are doing when we introduce, for example $\frac{2}{3}$. We separate the (unit) region into 3 congruent parts. Then we shade 2 of these parts. Similarly, in $\frac{3}{2}$, we separate each (*unit*) region into 2 congruent regions, and shade 3 parts. In using regions to represent a number like $\frac{3}{2}$, we *must* emphasize the fact that we are thinking of $\frac{3}{2}$ of a *unit* region, as in Fig. C.

(b) *Using the number line*. The steps used with regions can be carried out on the number line. It is easy to see that this is a very practical thing to do. If we have a ruler marked only in inches, we cannot make certain types of useful measurements. We need to have points between the unit intervals, and we would like to have numbers associated with these points.

The way we locate new points on the ruler parallels the procedure we followed with regions. We mark off each unit segment into congruent parts. We count off these parts. Thus, in order to locate the point corresponding to $\frac{2}{3}$, we must mark off the unit segment in 3 congruent parts. We then count off 2 of them. (Fig. D) If we have separated each unit interval in 2 congruent parts and counted off 3 of them, we have located the point which we would associate with $\frac{3}{2}$. (Fig. E)

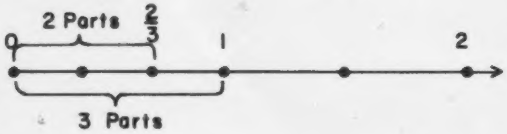


Fig. D

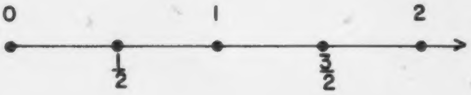


Fig. E

Once we have this construction in mind, we see that all such numbers as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{2}{4}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{11}{8}$ can be associated with particular points on the number line. To locate $\frac{11}{8}$, for example, we mark the unit segments into 8 congruent segments. (Fig. F)

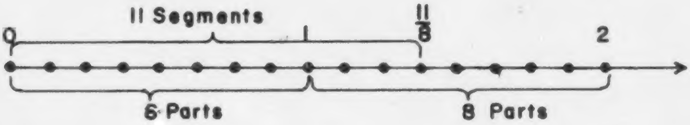


Fig. F

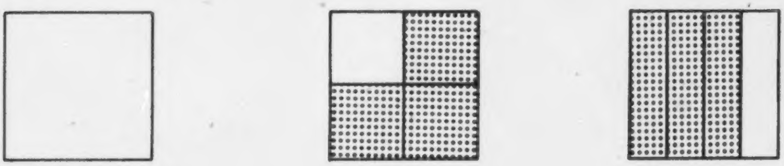
(c) *Numerals for pairs of numbers*. Suppose that we consider a pair of counting numbers such as 11 and 8 where 11 is the first number and 8 is the second number. We can make a symbol, writing the name of the first number of the pair above the line and that of the second below. Thus for the pair of numbers, 11 and 8, our symbol would be $\frac{11}{8}$. If we had thought of 8 as the first number of the pair and 11 as the second, we would have said the pair 8 and 11, and the symbol would have been $\frac{8}{11}$. For the numbers 3 and 4, the symbol would be $\frac{3}{4}$. For the numbers 4 and 3, the symbol would be $\frac{4}{3}$.

With the symbol described in the preceding paragraph, we can associate a point on the number line. The second number tells into how many congruent segments to separate each unit segment. The first number tells how many segments to count off.

We also can associate each of our symbols with a shaded region as in Fig. A, B, and C. The second number tells us into how many congruent parts we must separate each unit region. The first number tells us how many of these parts to shade.

For young children, regions are easier to see and to work with than segments. However, the number line has one strong advantage. For example, we associate a number as $\frac{3}{4}$, with exactly one point on the number line. The number line also gives an unambiguous picture for numbers like $\frac{3}{2}$ and $\frac{7}{2}$. A region corresponding to $\frac{3}{4}$ is less precisely defined in that regions with the same measure need not be identical or even congruent.

In Fig. G, we can see that each shaded region is $\frac{3}{4}$ of a unit square. Recognizing that *both* shaded regions have $\frac{3}{4}$ sq. units is indeed one part of the area concept.



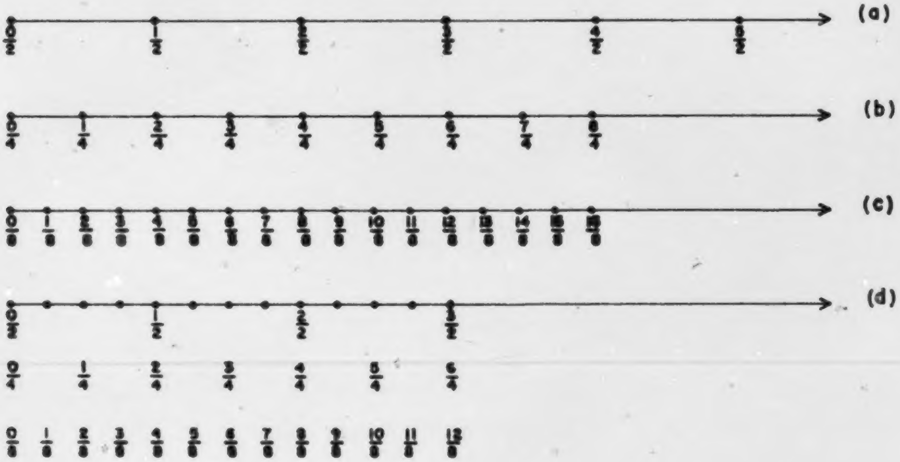
Unit square

Figure G

When we match numbers with points on the number line, we work with segments that begin at 0. For this reason, though the number line is less intuitive at early stages, it is well to use it as soon as possible.

Meaning of Rational Number

The diagrams Fig. H, (a), (b), (c), show a number line on which we have located points corresponding to $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc. and a number line on which we have located points corresponding to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, etc. Also shown, is a number line with $\frac{1}{8}$, $\frac{2}{8}$, etc. As we look at these lines, we see that it seems very natural to think of $\frac{0}{2}$ as being associated with the 0 point. We are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.


(a)
0/2 1/2 2/2 3/2 4/2 5/2
(b)
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
(c)
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8
(d)
0/2 1/2 2/2 3/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8

Now let us put our diagrams (a), (b), (c) together. In other words, let us carry out on a single line (d) the process for locating all the points.

When we do this, we see that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are all associated with the same point. In the same way, $\frac{3}{4}$ and $\frac{6}{8}$ are associated with the same point.

Now we are ready to explain more precisely what we mean by fraction and by rational number. Let us agree to call the symbols we have been using fractions. A fraction, then, is a symbol associated with a pair of numbers. The first number of the pair is called the numerator and the second number is called the denominator. So far, we have used only those fractions in which the numerator of the number pair is a whole number (0, 1, 2, ...), and the denominator is a counting number (1, 2, 3, ...).

Each fraction can be used to locate a point on the number line. To each point located by a fraction there corresponds a rational number. Thus, a fraction names the rational number. For example, if we are told the fraction $\frac{3}{10}$, we can locate a point that corresponds to it on the number line. $\frac{3}{10}$ is the name of the rational number associated with this point. This point, however, can also be located by means of other fractions, such as $\frac{6}{20}$ and $\frac{9}{30}$. Thus, $\frac{6}{20}$ and $\frac{9}{30}$ also are names for the rational number named by $\frac{3}{10}$ since they are associated with the same point. Rational numbers, then, are named by fractions of the type we have been discussing. To each point on the number line that can be located by a fraction, there corresponds a non-negative rational number.

A very unusual child might wonder whether every point on the number line can be located by a fraction of the kind we have described. We must answer "No". There are numbers--π being one of them and $\sqrt{2}$ being another--that have no fraction names of the sort we have described. Introducing such irrational numbers is deferred until the seventh and eighth grades.

The Whole Numbers As Rational Numbers

Our pattern for matching fractions with points on the number line can be used with these fractions: $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.

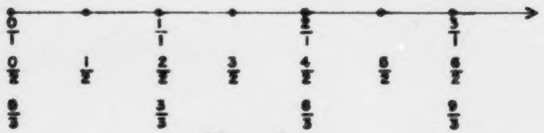


Fig. I

On the number line we see (Fig. I) that we matched $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$ with the same point. We note that this point is also matched with the counting number 1. Thus, to the same point corresponds

(1) the counting number 1

(2) the rational number named by $\frac{1}{1}$.

It seems that it would be a convenience to use the symbol 1 as still another name for the rational number named by $\frac{1}{1}$, $\frac{2}{2}$, etc. This would allow us to write $1 = \frac{2}{2}$, for example. In the same way, we would think of 5 as another name for the number named by $\frac{5}{1}$, $\frac{10}{2}$, etc.

We need at this point to be a little careful in our thinking. There is nothing illogical about using any symbol we like as a numeral. A problem does arise, however, when a single symbol has two meanings, because then we are in obvious danger that inconsistencies may result. For example, when we think of 2, 3, and 6 as counting numbers we are accustomed to writing $2 \times 3 = 6$. We will eventually define the product of two rational numbers, and we would be in serious trouble if the product of the rational numbers named by 2 and 3 were anything but the rational number named by 6.

However, using 0, 1, 2, 3, etc., as names for rational numbers never leads us into any inconsistency. For all the purposes of arithmetic--that is, for finding sums, products, etc., and for comparing sizes, we get names for whole numbers or names for rational numbers. In more sophisticated mathematical terms, we can say that the set of rational numbers contains a subset--those named by $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc.--isomorphic to the set of whole numbers, that is $\frac{0}{1}$, $\frac{1}{1}$, etc. behave just like whole numbers, 0, 1, etc.

It would be overambitious to attempt to formulate the idea of isomorphism precisely in our teaching. It is sufficient for our purposes to regard 0, 1, 2, etc., as names for rational numbers. It is appropriate to note, however, in connection with operations on rationals, that where the operations are applied to numbers like $\frac{1}{1}$, $\frac{2}{1}$ they lead to results already known from experience with whole numbers.

Identifying Fractions That Name The Same Rational Number

When we write $\frac{1}{2} = \frac{3}{6}$, we are saying "$\frac{1}{2}$ and $\frac{3}{6}$ are names for the same number."

(a) Using physical models. The truth of the sentence $\frac{1}{2} = \frac{3}{6}$ can be discovered by concrete experience. In Fig. J, for example, we have first separated our unit region into two congruent regions. We have then separated each of these parts further into 3 congruent regions as shown in the second drawing. The second unit square is thus separated into 2 × 3, or 6 parts. Shading 1 part in the first drawing is equivalent to shading 1 × 3, or 3 parts in the second. We thus recognize that $\frac{1}{2} = \frac{1 \times 3}{2 \times 3}$.

Shading $\frac{1}{2}$ and $\frac{3}{6}$ of a region.

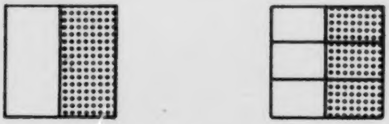


Fig. J

Again, our analysis of regions follows a pattern that can be applied on the number line. Let us consider $\frac{1}{2}$ and $\frac{4}{8}$.

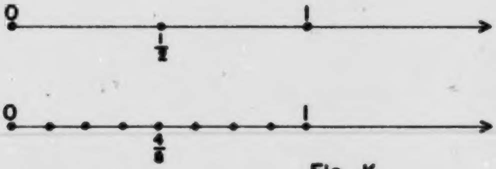


Fig. K

In locating $\frac{1}{2}$ on the number line, (Fig. K) we separate the unit interval into 2 congruent segments. In locating $\frac{4}{8}$, we separate it into 8 congruent segments. We can do this by first separating into 2 parts and then separating each of these 2 segments into 4 segments. This process yields (2 × 4) congruent segments. Taking 1 of 2 congruent parts thus leads to the same point as taking 4 of 8 congruent parts:

$$\frac{1}{2} = \frac{1 \times 4}{2 \times 4}$$

In other words, when we multiply the numerator and denominator of $\frac{1}{2}$ by the same counting number, we can visualize the result using the number line. We have subdivided our $\frac{1}{2}$ intervals into a number of congruent parts.

After many such experiences, children should be able to make a picture to explain this type of relationship. For example, region and number line pictures for $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$ are shown in Fig. L.

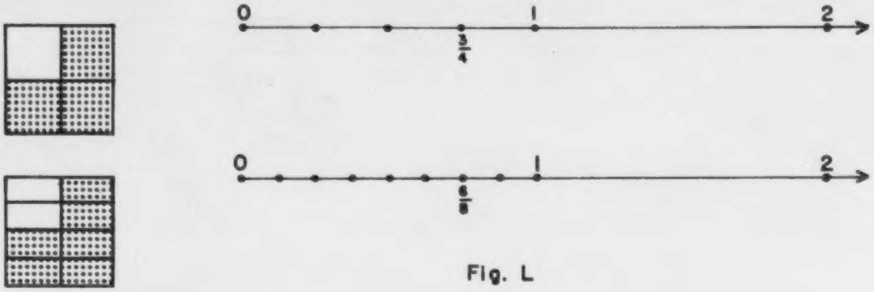


Fig. L

Each $\frac{1}{4}$ part (region or interval) is subdivided into 2 congruent parts; hence $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$.

(b) Using numerators and denominators. In a discussion about two fractions naming the same number, it may appear startling to emphasize multiplying numerator and denominator by the same counting number. We usually think about finding the simplest fraction name if we can. We think, then, $\frac{4}{8} = \frac{1}{2}$. But, of course "=" means "names the same number." Seeing $\frac{1}{2} = \frac{4}{8}$, we can think, $\frac{4}{8} = \frac{1}{2}$, and this will be particularly easy if the "names the same number" idea has been emphasized adequately.

Another familiar idea also is contained in what has been said. We often think about dividing numerator and denominator by the same counting number. For example, we think:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

This is easy to translate into a multiplicative statement, since multiplication and division are inverse operations: $6 \div 2 = 3$ means $3 \times 2 = 6$.

(c) Using factoring. The idea that multiplying the numerator and denominator of a fraction by a counting number gives a new fraction that names the same number as the original fraction is an idea very well suited to the discussion in the unit on factoring. To find a simpler name for $\frac{12}{15}$, we write:

$$\frac{12}{15} = \frac{2 \times 2 \times 3}{5 \times 3} = \frac{2 \times 2}{5} = \frac{4}{5}$$

Suppose we are thinking about two fractions. How will we decide whether or not they name the same number? There are two possibilities.

Rule (1). It may be that for such fractions as $\frac{1}{2}$ and $\frac{2}{4}$, one fraction is obtained by multiplying the numerator and denominator of the other by a counting number. In other words, it may be that we can picture the fractions as was just done. Since $\frac{2}{4} = \frac{2 \times 1}{2 \times 2}$, $\frac{2}{4}$ and $\frac{1}{2}$ belong to the same set--thus name the same number.

Rule (2). It may be that, we cannot use Rule 1 directly. For example, $\frac{2}{4}$ and $\frac{3}{6}$ cannot be compared directly by Rule 1. However, we can use Rule 1 to see that $\frac{2}{4} = \frac{1}{2}$ and $\frac{3}{6} = \frac{1}{2}$, and in this way, we see that $\frac{2}{4}$ and $\frac{3}{6}$ name the same number.

Notice that in comparing $\frac{2}{4}$ and $\frac{3}{6}$, we might have used Rule 1 and 2 in a different way. We might have recognized that:

$$\frac{2}{4} = \frac{2 \times 3}{4 \times 3} = \frac{6}{12} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 2}{6 \times 2} = \frac{6}{12}.$$

or we might have said:

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} = \frac{12}{24} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4} = \frac{12}{24}.$$

In the latter example, we have renamed $\frac{2}{4}$ and $\frac{3}{6}$, using fractions with denominator 4×6. Of course, we recognize that $4 \times 6 = 6 \times 4$. (Commutative Property)

In our example, we see that 24 is a common denominator for $\frac{2}{4}$ and $\frac{3}{6}$, though it is not the least common denominator. Nevertheless, one common denominator for two fractions is always the product of the two denominators.

(d) *A special test*. Let us now consider a special test for two fractions that name the same rational number. In our last example we used 6×4 as the common denominator for $\frac{2}{4}$ and $\frac{3}{6}$. Thus we had

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4}.$$

We could say: It is true that $\frac{2}{4} = \frac{3}{6}$, because the two resulting numerators--2×6 and 3×4--are equal, and the denominators are equal.

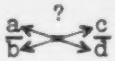
In other words, to test whether $\frac{2}{4} = \frac{3}{6}$, it is only necessary--*once you have understood the reasoning*--to test whether $2 \times 6 = 3 \times 4$. And this last number sentence is true!

In the same way, we can test whether $\frac{9}{15} = \frac{8}{40}$ by testing whether $9 \times 40 = 8 \times 15$. They do! When we do this, we are thinking:

$$\frac{9}{15} = \frac{9 \times 40}{15 \times 40} \qquad \text{and} \qquad \frac{8}{40} = \frac{8 \times 15}{40 \times 15}$$

This is an example of what is sometimes called "cross product rule." It is very useful in solving proportions. (Sometimes it is stated: The product of the means equals the product of the extremes.)

The rule states: To test whether two fractions $\frac{a}{b}$ and $\frac{c}{d}$ name the same number, we need only test whether $a \times d = b \times c$. That is,



This rule is important for later applications in mathematics such as similar triangles. In advanced texts on algebra, it is sometimes used as a way of *defining* rational numbers. That is, an advanced text might say: "A rational number is a set of symbols like $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$. Two symbols, $\frac{a}{b}$ and $\frac{c}{d}$, belong to the same set if $a \times d = b \times c$."

What we have done amounts to the same thing, but is developed more intuitively. For teaching purposes, the "multiply numerator and denominator by the same counting number" idea conveyed by Rule 1 can be visualized more easily than can the "cross product" rule.

It would certainly not be our intention to insist that children learn Rules 1 and 2 formally. However, these rules summarize an experience that is appropriate for children. We can form a chain of fractions that name the same number,

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \ . \ . \ . \ .$$

Each fraction is formed by multiplying the numerator and denominator of the preceding one by 2. We can visualize this as subdividing repeatedly a segment or a region. (Rule 1). We can form a second chain beginning with $\frac{1}{2} = \frac{3}{6} = \frac{9}{18} \ . \ . \ .$ We can then understand that it is possible to pick out any numeral from one chain and equate it with any numeral from the other, which is just what Rule 2 says.

Meaning of Rational Number - Summary

Let us summarize how far we have progressed in our development of the rational numbers.

(1) We regard a symbol like one of the following as naming a rational number:

$$\frac{3}{8}, \ \frac{0}{5}, \ \frac{7}{6}, \ 6, \ \frac{4}{3}, \ \frac{6}{4}, \ 1, \ \frac{5}{5}$$

(2) We know how to associate each such symbol with a point on the number line.

(3) We know that the same rational number may have many names that are fractions. Thus, $\frac{6}{4}$ and $\frac{3}{2}$ are fraction names for the same number.

(4) We know that when we have a rational number named by a fraction, we can multiply the numerator and denominator of the fraction by the same counting number to obtain a new fraction name for the same rational number.

(5) We know that in comparing two rational numbers it is useful to use fraction names that have the same denominators. We know, too, that for any two rational numbers, we can always find fraction names of this sort.

Thus far we have not stressed what is often called, in traditional language, "reducing fractions." to "reduce" $\frac{6}{8}$, for example, is simply to name it with the name using the smallest possible numbers for the numerator and the denominator. Since 2 is a factor both of 6 and 8, we see that

$$\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4}$$

We have applied our general idea that "multiplying numerator and denominator by the same counting number" gives a new name for the same number. We can call $\frac{3}{4}$ the simplest name for the rational number it names.

We would say that we have found, in $\frac{3}{4}$, the simplest name for the rational number named by $\frac{6}{8}$. This is more precise than saying we have "reduced" $\frac{6}{8}$, since we have not made the rational number named by $\frac{6}{8}$ any smaller. We have used another pair of numbers to rename it.

(6) We know, also, that 2 and $\frac{2}{1}$ name the same number. We thus regard the set of whole numbers as a subset of the set of rational numbers. Any number in this subset has a fraction name with denominator 1. ($\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc. belong to this subset.) 2 is a name for a rational number which is a whole number. 2 is not a fraction name for this number, but the number has fraction names $\frac{2}{1}$, $\frac{4}{2}$, etc.

At this point, it seems reasonable to use "number" for rational numbers where the meaning is clear. We may ask for the number of inches or measure of a stick, or the number of hours in a school day.

(7) We can agree to speak of the number $\frac{2}{3}$, to avoid the wordiness of "number named by $\frac{2}{3}$." Thus, we might say that the number $\frac{2}{3}$ is greater than the number $\frac{1}{2}$ (as we can verify easily on the number line). This would be preferable to saying that "the fraction $\frac{2}{3}$ is greater than the fraction $\frac{1}{2}$," because we do not mean that one name is greater than another.

(8) We should not say that 3 is the denominator of the number $\frac{2}{3}$, because the same number has other names (like $\frac{4}{6}$) with different denominators. 3 is rather the denominator of the fraction $\frac{2}{3}$.

(9) We have seen that the idea of rational number is relevant both to regions and line segments. We will see soon how it relates to certain problems involving sets.

Now we might introduce some decimals. The numeral, .1, for example, is another name for $\frac{1}{10}$. However, we can explain a numeral like 1.7 more easily when we have developed the idea of adding rational numbers.

Operations on Rational Numbers

Now let us consider the operations of arithmetic for rational numbers. For each, our treatment will be based on three considerations:

(1) The idea of rational number grows out of ideas about regions and the number line. Similarly, each operation on rational numbers can be "visualized" in terms of regions or the number line. Indeed, this is how people originally formed the ideas of sum, product, etc. of rational numbers. Each operation was introduced to fit a useful physical situation and not as a way of supplying problems for arithmetic textbooks.

(2) We recall that some rational numbers are whole numbers. So, we want our rules of operation to be consistent with what we already know about whole numbers.

(3) We must remember that the same rational number has many names. We will want to be sure that the result of an operation on two numbers does not depend on the special names we choose for them. For example, we want the sum of $\frac{1}{2}$ and $\frac{1}{3}$, to be the same number as the sum of $\frac{2}{4}$ and $\frac{2}{6}$.

These three ideas will guide us in defining the operations of addition, subtraction, multiplication, and division of rational numbers.

Addition and Subtraction

As an illustration of addition, we might think of a road by which stand a house, a school and a store as shown in Fig. M. If it is $\frac{1}{2}$ mile from the house to the school, and $\frac{1}{4}$ mile from the school to the store, then we can see that the distance from the house to the store is $\frac{3}{4}$ mile.

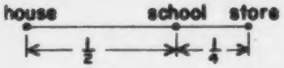


Fig. M

From such examples we can see the utility of defining addition of rational numbers by using the number line. To find the sum of $\frac{2}{5}$ and $\frac{4}{5}$ we would proceed as in Figure N.

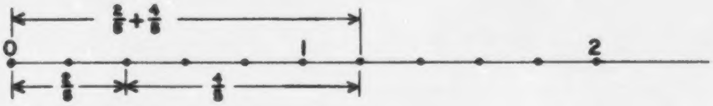


Fig. N

Using a ruler, we can locate the point on the number line corresponding to the sum of any two rational numbers. For example, with appropriate rulers, a child can locate the point that corresponds to the sum of $\frac{4}{5}$ and $\frac{3}{8}$. But a child would also like to know that the point for the sum located with a ruler is one for which he can find a fraction name--a fraction that names a rational number. Of course, one name for the sum of $\frac{4}{5}$ and $\frac{3}{8}$ is $\frac{4}{5} + \frac{3}{8}$, but what is the single fraction that names this number? Also, he is interested in knowing whether or not the set of rational numbers is closed under addition, since he knows that this is true for the whole numbers.

Using the number line, it is evident that the sum of $\frac{2}{4}$ and $\frac{3}{4}$ is $\frac{5}{4}$. This suggests a way to find the sum of two rational numbers that are named by fractions with the same denominator. For such fractions, we simply add the numerators. Thus, $\frac{3}{8} + \frac{4}{8} = \frac{3 + 4}{8} = \frac{7}{8}$. This definition matches the idea of joining two line segments.

But we are not finished! For suppose that we want to add $\frac{2}{3}$ and $\frac{1}{4}$. We know that there are many other names for the number named by $\frac{2}{3}$. Some are:

$$\frac{4}{6}, \frac{6}{9}, \frac{8}{12}.$$

Likewise, there are many names for the number named by $\frac{1}{4}$. They include:

$$\frac{2}{8}, \frac{3}{12}, \frac{4}{16}.$$

In order to find the sum of these two rational numbers we simply look for a pair of names with the same denominator--that is, with a common denominator. Having found them, we apply our simple process of adding numerators.

Thus we can write a fraction name for the sum of two rational numbers if we can write fraction names with the same denominators for the numbers. This we can always do, for to find the common denominator of two fractions, we need only to find the product of their denominators.

This provides a good argument as to why

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}.$$

It is clear that $\frac{8}{12} + \frac{3}{12} = \frac{11}{12}$. If the idea that the same number has many names makes any sense at all, it must be true that

$$\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12}.$$

Suppose that in our example we had used a different common denominator, as 24. Would we get a different result? We see that we would not for:

$$\begin{array}{cccc}
\frac{2}{3} & = \frac{2 \times 8}{3 \times 8} & = & \frac{16}{24} \\
\frac{1}{4} & = \frac{1 \times 6}{4 \times 6} & = & \frac{6}{24} \\
\hline
\frac{2}{3} + \frac{1}{4} & & = & \frac{22}{24} \\
\text{and } \frac{22}{24} & & = & \frac{11}{12}.
\end{array}$$

Little needs be said here about subtraction. Using the number line we can visualize $\frac{3}{4} - \frac{2}{3}$ as in Figure P.

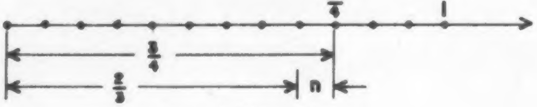


Fig. P

Thus we can define $\frac{3}{4} - \frac{2}{3}$ as the number n such that $\frac{2}{3} + n = \frac{3}{4}$. Again, skillfully chosen names lead at once to the solution:

$$\frac{8}{12} + n = \frac{9}{12}$$

$$n = \frac{1}{12}$$

Properties of Addition for Rational Numbers

Our rule for adding rational numbers has some by-products worth noting.

We can see, for one thing, that addition of rational numbers is commutative. Our number line diagram illustrates this. In Fig. 0 we see the diagram for $\frac{1}{5} + \frac{2}{5}$ and for $\frac{2}{5} + \frac{1}{5}$.

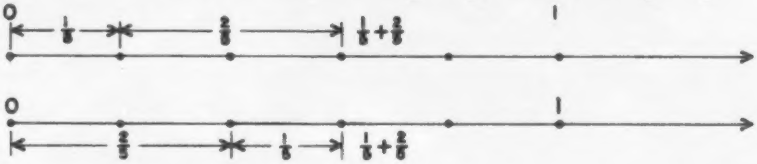


Fig. 0

The commutative property also can be explained in another way.

$$\frac{1}{5} + \frac{2}{5} = \frac{1 + 2}{5} \quad \text{and} \quad \frac{2}{5} + \frac{1}{5} = \frac{2 + 1}{5}$$

We know that $1 + 2 = 2 + 1$, so we see that $\frac{1}{5} + \frac{2}{5} = \frac{2}{5} + \frac{1}{5}$. In general, to add rational numbers named by fractions with the same denominator we simply add numerators. Adding numerators involves adding whole numbers. We know that addition of whole numbers is commutative. This leads us to conclude that addition of rational numbers is also commutative.

We can use this type of discussion or the number line diagram to see that addition of rational numbers is also associative.

Here is another interesting property of addition of rational numbers. We recall that $\frac{0}{1}$, $\frac{0}{2}$, $\frac{0}{3}$, etc. are all names for 0. Thus $0 + \frac{3}{4} = \frac{0}{4} + \frac{3}{4} = \frac{0 + 3}{4} = \frac{3}{4}$. In general we see that the sum of 0 and any rational number is the number.

Similarly we recall, for example, that $\frac{2}{1}$ and $\frac{3}{1}$ are fraction names for 2 and 3 respectively. Thus

$$2 + 3 = \frac{2}{1} + \frac{3}{1} = \frac{2+3}{1} = \frac{5}{1} = 5$$

as we would expect (and hope).

Addition of rational numbers is not difficult to understand, once the idea that the same rational number has many different fraction names has been well established. The technique of computing sums of rational numbers written with fraction names is in essence a matter of finding common denominators. This is essentially the problem of the least common multiple and thus is a problem about whole numbers.

Addition of Rational Numbers Using Other Numerals

Often it is convenient to use numerals other than fractions to find the sum of two rational numbers. Those commonly used are mixed forms and decimals.

The first kind of numeral can be easily understood once addition has been explained. We can see with line segments that

$$2 + \frac{1}{3}$$

is a rational number, and it is also easy to see that $\frac{7}{3}$ is another name for this same number. Similarly, $\frac{15}{4} = 3\frac{3}{4}$. Indeed, these ideas can be introduced before any formal mechanism for adding two rational numbers named by fractions has been developed, because the idea that $2 + \frac{1}{3} = \frac{7}{3}$ goes back to the number line idea of sum. To adopt the convention of writing $2\frac{1}{3}$ as an abbreviation for $2 + \frac{1}{3}$ is then easy, and we may use a numeral like $2\frac{1}{3}$ as a name for a rational number. It is these we call a numeral in mixed form.

The use of decimals is still another convention for naming rational numbers. For example, 3.2 names a rational number; other names for this number are

$$3 + \frac{2}{10},\ 3\frac{2}{10},\ 3\frac{1}{5},\ \frac{16}{5},\ \frac{32}{10},\ \frac{320}{100}.$$

Of these, $\frac{16}{5}$, $\frac{32}{10}$, and $\frac{320}{100}$ are fraction names while $3\frac{2}{10}$ and $3\frac{1}{5}$ are mixed forms.

The methods for computing with decimals are direct outcomes of their meaning. For example, to compute 3.4 + 1.7, we may proceed as follows:

$$\begin{array}{r} 3.4 = 3 + \frac{4}{10} \\ 1.7 = 1 + \frac{7}{10} \\ \hline 4 + \frac{11}{10} = 5 + \frac{1}{10}. \end{array}$$

Hence 3.4 + 1.7 = 5.1.

We want the child to develop a more efficient short-cut procedure for finding such a sum. However, the understanding of the procedure can be carried back, as shown, to the knowledge he already has about adding numbers with names in fraction or mixed form.

In a similar way, the procedures for subtracting, multiplying and dividing numbers named by decimals can be understood in terms of the same operations applied to numbers named by fractions.

Multiplication

By the time the child is ready to find the product of two rational numbers such as $\frac{2}{3}$ and $\frac{3}{4}$, he has already had a number of experiences in understanding and computing products of whole numbers.

He has seen 3 × 2 in terms of a rectangular array. He can recognize the arrangement in Fig. Q as showing 3 groups of objects with 2 objects in each group.

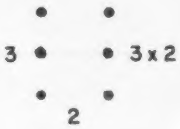


Fig. Q

Also, he has seen 3 × 2 in terms of line segments. (Fig. R) that is, as a union of 3 two-unit segments. Furthermore, he has interpreted

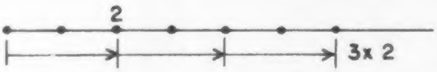


Fig. R

Figure R in terms of travel along a line. For example, if he rows a boat 2 miles an hour across a lake, then in 3 hours he rows 6 miles.

Finally, 3 × 2 can be related to areas as in Figure S. Thus, a child has seen that the operation of multiplication can be applied to many physical models. He has related several physical situations to a single number operation.

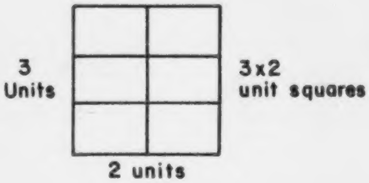


Fig. S

The "Rectangular Region" Model

May we remind you that the idea of multiplying $\frac{2}{3}$ and $\frac{1}{5}$ was *not* invented for the purpose of writing arithmetic books. Instead, people found some applications in which the numbers $\frac{2}{3}$ and $\frac{1}{5}$ appeared and *also* $\frac{2}{15}$ appeared. For instance, in Fig. T we see a unit square separated into 15 congruent rectangles. The measure of the shaded region is $\frac{2}{15}$ square units. On the other hand, we have already used the operation of *multiplication* to compute areas of rectangles having dimensions that are *whole* numbers. Hence it is natural to say: Let us call $\frac{2}{15}$ the *product* of $\frac{2}{3}$ and $\frac{1}{5}$ and write $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$.

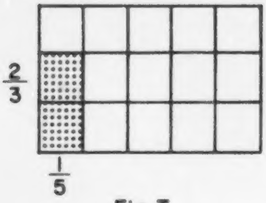


Fig.T

Logically, $\frac{2}{3} \times \frac{1}{5}$ is a meaningless symbol until we define it. It could mean anything we choose. Our choice of $\frac{2}{15}$ for a meaning seems, however, a useful one, and indeed it is.

Yet, children need many more examples before they can see the general rule that in multiplying rational numbers named by fractions we multiply the numerators and multiply the denominators.

We should recognize that although the formal introduction of $\frac{2}{3} \times \frac{1}{5}$ is deferred until the sixth grade the development is anticipated by many earlier experiences. Among them are: the identification of a fraction with a region and the various steps in finding the measure of a region.

The "Number Line" Model

The product of two whole numbers also can be visualized on the number line. A few natural generalizations to products of rational numbers can be made from these kinds of experiences.

For example, if we can think of 3×2 as illustrated by Figure U, (a), then it is natural to identify $3 \times \frac{1}{4}$ with the situation pictured in Figure U, (b).

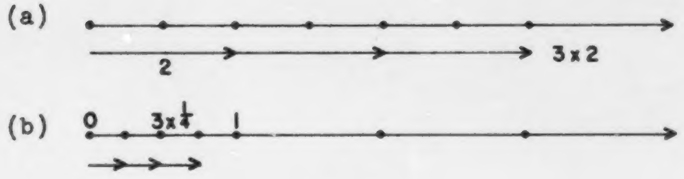


Fig. U

In the same way we can identify, for example, $3 \times \frac{2}{7}$ with

$$\frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \frac{6}{7}.$$

Again, if a man walks 4 miles each hour, then (2×4) miles is the distance he walks in 2 hours. (Fig. V) Once more it is also natural to relate $\frac{1}{2} \times 4$ with a distance he travels--this time with his distance in $\frac{1}{2}$ hour.

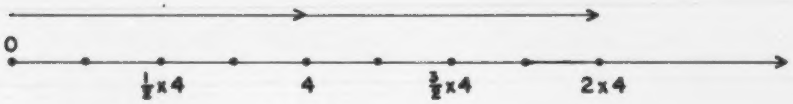


Fig. V

Suppose, now, that a turtle travels $\frac{1}{5}$ mile in an hour. In 2 hours, it travels $2 \times \frac{1}{5}$, or $\frac{2}{5}$ miles. We identify the product $\frac{2}{3} \times \frac{1}{5}$ with the distance it travels in $\frac{2}{3}$ of an hour.

Fig. W diagrams the turtle's travels.

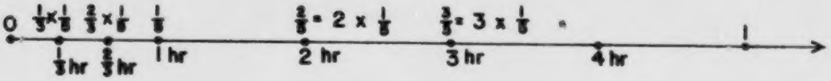

0
$\frac{1}{3} \times \frac{1}{5}$
$\frac{2}{3} \times \frac{1}{5}$
$\frac{1}{5}$
$\frac{2}{5} = 2 \times \frac{1}{5}$
$\frac{3}{5} = 3 \times \frac{1}{5}$
1
$\frac{1}{3}$ hr
$\frac{2}{3}$ hr
1 hr
2 hr
3 hr
4 hr

Fig. W

We locate $\frac{2}{3}$ of $\frac{1}{5}$ on the number line by locating $\frac{1}{5}$, cutting the $\frac{1}{5}$ segment into 3 congruent segments, and counting off two of them, as in Figure X.

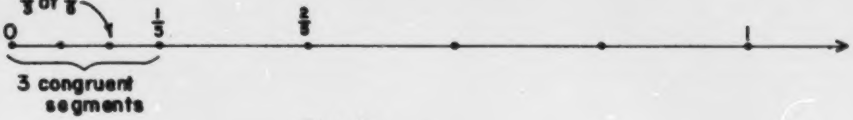

$\frac{2}{3}$ of $\frac{1}{5}$
0
$\frac{1}{5}$
$\frac{2}{5}$
1
3 congruent segments

Fig. X

More specifically, we first cut the unit segment into 5 congruent segments. Then each of these is cut into 3 congruent segments. We thus have 3 × 5 segments. We counted 2 × 1 of them. We see that $\frac{2}{3}$ of $\frac{1}{5}$ is associated with the point $\frac{2 \times 1}{3 \times 5}$.

We had two numbers: $\frac{2}{3}$ and $\frac{1}{5}$. When we talk about $\frac{2}{3}$ of $\frac{1}{5}$ we are explaining a situation in which we have a pair of numbers ($\frac{2}{3}$ and $\frac{1}{5}$) associated with a third ($\frac{2}{15}$). We have, in short, an operation; it is natural to see whether it is an operation we know. We find that it is.

We already had agreed, using the rectangle model, that

$$\frac{2}{3} \times \frac{1}{5} = \frac{2 \times 1}{3 \times 5} = \frac{2}{15}.$$

Hence, we now see that we can identify $\frac{2}{3}$ of $\frac{1}{5}$ with $\frac{2}{3} \times \frac{1}{5}$.

Moreover, we notice that if we use the idea of travel on the number line (Fig. W) it is again natural to identify $\frac{2}{3} \times \frac{1}{5}$ with $\frac{2}{3}$ of $\frac{1}{5}$.

The "Sets of Objects" Model

We speak, in everyday usage, of $\frac{2}{3}$ of a dozen of eggs. We visualize this as the result of separating a finite set of 12 objects into 3 subsets each with the same number of objects and then uniting 2 of the subsets. The relation between this concept and that involving a 12-inch segment can be seen from Figure Y.

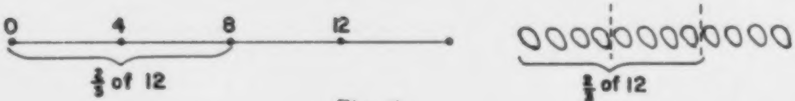


Fig. Y

We sometimes use such examples with very young children to emphasize the idea of $\frac{2}{3}$. But this is a little misleading, for we should note that $\frac{2}{3}$ of 12 is again a situation involving two numbers, $\frac{2}{3}$ and 12. Again we can verify that $\frac{2}{3}$ of 12 is computed by finding $\frac{2}{3} \times 12$.

Summary

From the standpoint of defining the operation of multiplication for rational numbers, it would be entirely sufficient to use one interpretation. However, because products of rational numbers are used in many types of problem situations the child ought to recognize that the definition does fit the needs of each.

$\frac{2}{3} \times \frac{1}{4}$ can be visualized as:

(1) the area in square inches of a rectangle with length $\frac{2}{3}$ in. and width $\frac{1}{4}$ in.

(2) the length of a line segment formed by taking $\frac{2}{3}$ of a $\frac{1}{4}$ inch segment.

Out of the number line model come many problem situations. For example, if a car travels $\frac{1}{4}$ mile per minute, it travels $\frac{2}{3} \times \frac{1}{4}$ miles in $\frac{2}{3}$ minute.

Moreover, $\frac{2}{3} \times 12$ can be interpreted in the ways noted and also can be related to finite sets. We use $\frac{2}{3} \times 12$ where we want to find the number of eggs in $\frac{2}{3}$ of a dozen.

Mathematics is powerful because a single mathematical idea (like $\frac{2}{3} \times \frac{1}{4}$) often has many applications. Children can fully understand a product like $\frac{2}{3} \times \frac{1}{4}$ only when they have had experiences with several applications.

We should observe that our definition of the product of two rational numbers is consistent with what we already know about whole numbers. We know that 2×3, for example, is another name for 6. All is well, for the product of $\frac{2}{1}$ and $\frac{3}{1}$, as computed by our definition, is $\frac{6}{1}$, and $\frac{6}{1}$ names the same number as 6. We note, too, that our definition leads us to $3 \times \frac{1}{4} = \frac{3}{1} \times \frac{1}{4} = \frac{3}{4}$, as was anticipated earlier.

We should notice, too, that although our method for finding the product is expressed in terms of specific names for the factors, the product is not changed if we change the names. For example, $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$. Renaming $\frac{1}{2}$ and $\frac{3}{4}$ we have

$$\frac{3}{6} \times \frac{6}{8} = \frac{18}{48}.$$

$\frac{18}{48}$ is another name for $\frac{3}{8}$.

Properties of Multiplication of Rational Numbers

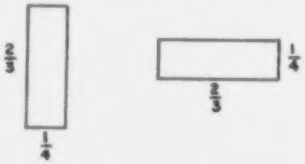
Again our definition has convenient by-products. For, we observe that the associative and commutative properties hold multiplication of rational numbers. They hold here as a direct result of the same properties for multiplication of whole numbers.

The following example shows how we may explain the commutative property of multiplication.

Our rule for multiplication tells us that $\frac{2}{3} \times \frac{1}{4} = \frac{2 \times 1}{3 \times 4}$. Our rule tells us also that $\frac{1}{4} \times \frac{2}{3} = \frac{1 \times 2}{4 \times 3}$. We know, however, that $1 \times 2 = 2 \times 1$ and $3 \times 4 = 4 \times 3$. These facts are instances of the commutative property of multiplication of whole numbers. Hence we see: $\frac{2}{3} \times \frac{1}{4} = \frac{1}{4} \times \frac{2}{3}$.

We also can visualize this property using rectangular regions as in Figure Z.

Fig. Z



The *associative property of multiplication* holds, in essence, because it holds *for whole numbers*. From the associative property, we can compute $\frac{1}{2}(4 \times 5)$ as either $\frac{1}{2} \times (4 \times 5)$ or $(\frac{1}{2} \times 4) \times 5$--a fact which is sometimes helpful in using the formula for the area of a triangle.

We observe, too, an interesting *multiplication property of 0*. Our rule for multiplying two rational numbers named by fractions leads directly to the conclusion that the product of 0 and any rational number is 0:

$$0 \times \frac{2}{3} = \frac{0}{1} \times \frac{2}{3} = \frac{0 \times 2}{1 \times 3} = \frac{0}{3} = 0.$$

Of interest, too, is the *multiplication property of 1*. It is easy to see that the product of 1 and any number is the number:

$$1 \times \frac{2}{3} = \frac{1}{1} \times \frac{2}{3} = \frac{1 \times 2}{1 \times 3} = \frac{2}{3}.$$

Again this is a direct result of the same property of whole numbers. Now 1, of course, has many names. One of them, for example, is $\frac{3}{3}$. When we multiply $\frac{4}{5}$ by 1 we can write

$$1 \times \frac{4}{5} = \frac{3}{3} \times \frac{4}{5} = \frac{3 \times 4}{3 \times 5} = \frac{12}{15} = \frac{4}{5}.$$

This result shows that multiplying the numerator and denominator of a fraction by the same counting number is equivalent to multiplying 1 by the number named by the fraction.

Finally, the *distributive property* for rational numbers is an outcome of our definition. The distributive property tells us that, for example, $\frac{2}{3} \times (\frac{1}{5} + \frac{2}{5}) = (\frac{2}{3} \times \frac{1}{5}) + (\frac{2}{5} \times \frac{2}{5})$. Our area picture helps us to understand this easily (Fig. AA) The smaller rectangles have areas $\frac{2}{3} \times \frac{1}{5}$ and $\frac{2}{3} \times \frac{2}{5}$. The area of their union is $\frac{2}{3} \times (\frac{1}{5} + \frac{2}{5})$, or $\frac{2}{3} \times \frac{3}{5}$.

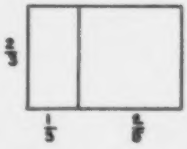


Fig. AA

The distributive property is useful in computing a product like $5 \times 3\frac{1}{2}$. We can say:

$$5 \times 3\frac{1}{2} = (5 \times 3) + (5 \times \frac{1}{2}) = 15 + \frac{5}{2} = 17\frac{1}{2}.$$

We can also recognize that we have essentially applied the distributive property in writing

$$\frac{3}{4} = 3 \times \frac{1}{4} = (1 + 1 + 1) \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}.$$

Before leaving the topic of multiplication of rational numbers, we ought to notice that the product of certain pairs of rational numbers is 1. For example, the product of $\frac{2}{3}$ and $\frac{3}{2}$ is 1. The number $\frac{2}{3}$ is called the reciprocal of $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$. The reciprocal of a number is the number it must be multiplied by to give 1 as a product. Every rational number except 0 has exactly one reciprocal. When the number is named by a fraction, we can easily find the reciprocal by "turning the fraction upside down". Thus the reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$.

In particular, the reciprocal of the whole number 2 is $\frac{1}{2}$, which can be verified easily since $2 \times \frac{1}{2} = 1$.

0 has no reciprocal. For we know that the product of 0 and every rational number is 0. Hence there is no number we can multiply by 0 to give 1.

Division

In the rational number system, as in the counting numbers, we want to use division to answer questions of "what must we multiply". In a division situation we are given one factor and a product. Thus $\frac{5}{3} \div \frac{3}{4}$ is the number such that: $(\frac{5}{3} \div \frac{3}{4}) \times \frac{3}{4} = \frac{5}{3}$. In order to compute $\frac{5}{3} \div \frac{3}{4}$ we must solve:

$$\frac{3}{4} \times \frac{m}{n} = \frac{5}{3}$$

where m and n are counting numbers.

To acquire an understanding of the division process, children need many concrete experiences in its use. These experiences parallel those with multiplication, since division problems can be interpreted as problems in finding an appropriate multiplier. Thus typical problem situations include: 1) finding the width of a rectangle when the length and area are known;

2) finding what fractional part one set is of another;
3) finding the number of segments of given length that can be made by cutting a given segment.

The Idea of Reciprocal and Division

We already have seen that the product of any number and its reciprocal is 1. For example,

$$\frac{2}{5} \times \frac{5}{2} = \frac{2 \times 5}{5 \times 2} = 1.$$

We also know that $1 \div \frac{2}{5} = n$ means $\frac{2}{5} \times n = 1$. Thus, the reciprocal of $\frac{2}{5}$ (which is $\frac{5}{2}$) is the number by which one can multiply $\frac{2}{5}$ to obtain the product 1.

Now suppose we want to find the number m such that

$$\frac{2}{5} \times m = 3.$$

Since $1 \times 3 = 3$, we can write: $\frac{2}{5} \times m = (1 \times 3)$. But since $\frac{2}{5} \times \frac{5}{2} = 1$, we also can write: $\frac{2}{5} \times m = (\frac{2}{5} \times \frac{5}{2}) \times 3$. Using the associative property, we can write again: $\frac{2}{5} \times m = \frac{2}{5} \times (\frac{5}{2} \times 3.)$ We now see that $m = \frac{5}{2} \times 3$.

We can use the same reasoning to compute $\frac{5}{3} \div \frac{3}{4}$. We must solve:

$$\frac{3}{4} \times m = \frac{5}{3}.$$

We know that $\frac{3}{4} \times \frac{4}{3} = 1$ and that the product of a number and one is the number itself. So,

$$(\frac{3}{4} \times \frac{4}{3}) \times \frac{5}{3} = \frac{5}{3}$$

and

$$\frac{3}{4} \times (\frac{4}{3} \times \frac{5}{3}) = \frac{5}{3}.$$

Therefore, $m = \frac{4}{3} \times \frac{5}{3} = \frac{4 \times 5}{3 \times 3} = \frac{20}{9}$.

We note: To *divide* $\frac{5}{3}$ by $\frac{3}{4}$, we *multiply* $\frac{5}{3}$ by the *reciprocal* of $\frac{3}{4}$.

We have seen a way in which we can derive the general rule: To divide by a non-zero rational number, multiply by its reciprocal.

We see that for whole numbers our rule gives the results we would expect. For example,

$$6 \div 2 = \frac{6}{1} \div \frac{2}{1} = \frac{6}{1} \times \frac{1}{2} = \frac{6}{2} = 3.$$

In particular, the reciprocal of 1 is 1, since $1 \times 1 = 1$. Thus when we divide a rational number by 1 we multiply it by 1 and obtain the original number, as we would expect.

Our rule for division identifies dividing by a number with multiplying by its reciprocal. Thus when we wish to find $\frac{1}{3}$ of 18 we may use either multiplication by $\frac{1}{3}$ or division by 3.

We now see that: $\frac{3}{4} = 3 \times \frac{1}{4} = 3 \div 4$. One important interpretation of $\frac{3}{4}$ as the result of dividing 3 by 4 can be visualized using line segments. (Fig. BB)

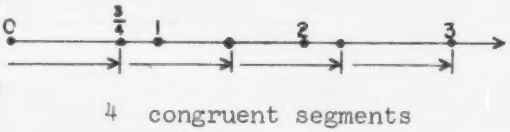


Fig. BB

$\frac{3}{4}$ can be seen, too as the answer to the question "How many 4's in 3 ?" More precisely, $\frac{3}{4}$ is the number by which we must multiply 4 to get 3.

Some texts for later grades define rational numbers by using the idea of division. That is, $\frac{3}{4}$ is defined from the outset as the number x satisfying $4x = 3$.

The set of counting numbers is not closed under division--that is, with only counting numbers at our disposal, we can not solve an equation like $4 \times n = 3$. But having introduced the set 7 rational numbers, we can always solve an equation of this type. We can divide any rational number by any number different from 0. Hence the set, made up of all the rational numbers except 0, is closed under division. We can interpret the extension of our idea of number from the counting numbers to the rational numbers as a successful effort to obtain a system of numbers that is closed under division.

It is an interesting paradox that now, having defined division by a rational number as multiplication by the reciprocal of the number, we could really get along without division entirely, since to divide by a number we can always multiply by its reciprocal. Later, we introduce the negative numbers to make a system closed under subtraction. Once we have done so, subtracting a number can be replaced by adding the opposite.

Fractions--A Symbol for a Pair of Rational Numbers

Thus far, we have restricted our use of fraction to that of being a symbol naming a pair of whole numbers. Let us now give meaning to symbols like $\frac{\frac{3}{2}}{\frac{6}{1}}$, $\frac{1.5}{.5}$, etc. in which the pairs of numbers are rational numbers instead of whole numbers.

We call that we already know $\frac{3}{4}$ and $3 \div 4$ are two names for the same number. That is, for $4 \times n = 3$, $n = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4}$. This suggests that we might say that the symbol $\frac{\frac{3}{2}}{6}$ will mean $\frac{3}{2} \div 6$, and the symbol $\frac{1.5}{.5}$ will mean $1.5 \div 5$.

A Definition

When we say, let $\frac{3}{2} \div 6 = \frac{\frac{3}{2}}{6}$, we are defining the meaning of those symbols which hitherto has had no meaning for us. There is nothing illogical with defining a new symbol in any way we like. However, simply assigning to $\frac{3}{2} \div 6$ the symbol $\frac{\frac{3}{2}}{6}$ does not permit us to treat this new symbol immediately as if it were a fraction of the kind with which we are familiar. For example, although we know that $\frac{3}{4} = \frac{2 \times 3}{2 \times 4}$ we cannot be certain that $\frac{\frac{3}{2}}{6} = \frac{2 \times \frac{3}{2}}{2 \times 6}$. Too, just because we know $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$, we cannot conclude (without argument) that $\frac{1}{.1} + \frac{1}{.1} = \frac{2}{.1}$.

In practice, children in the elementary school are unlikely to add or to multiply many numbers named by these new fractions. Yet, examples as $\frac{1.5}{.5}$ will be familiar when using decimal names in division of rational numbers. In later years, they can find solutions to such examples as $12\frac{1}{2}\%$ of 120 by solving

$$\frac{12.5}{100} = \frac{n}{120}.$$

Thus, it seems necessary to "know" if it is possible to multiply the numerator and denominator of $\frac{1.5}{.5}$ by 10 to obtain another name for the same number.

Again, let us make some observations about division. Does multiplying the dividend and divisor by the same number change the result? We observe

$$6 \div 2 = 3$$

$$(6 \times 2) \div (2 \times 2) = 3 \quad \text{or} \quad 12 \div 4 = 3$$

$$(6 \times \tfrac{1}{2}) \div (2 \times \tfrac{1}{2}) = 3 \quad \text{or} \quad \tfrac{6}{2} \div \tfrac{2}{2} = 3 \div 1 = 3.$$

We also need to be sure that when we multiply the numerator and denominator of a fraction, as, $\dfrac{\frac{5}{3}}{\frac{3}{4}}$, by the same number, we obtain a new fraction equal to the original one.

Does $\frac{5}{3} \div \frac{3}{4} = \dfrac{\frac{5}{3}}{\frac{3}{4}}$?

We know that $\frac{5}{3} \div \frac{3}{4} = \frac{3}{4} \times n = \frac{5}{3}$ and $n = \frac{4}{3} \times \frac{5}{3} = \frac{20}{9}$.

Let us now multiply both numerator and denominator of the fraction $\dfrac{\frac{5}{3}}{\frac{3}{4}}$ by the same number $\frac{4}{3}$.

$$\frac{\frac{5}{3} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{5 \times 4}{3 \times 3}}{\frac{3 \times 4}{4 \times 3}}$$

But $\frac{3 \times 4}{4 \times 3} = \frac{12}{12} = 1$ and $\frac{5 \times 4}{3 \times 3} = \frac{20}{9}$.

Is $\dfrac{\frac{20}{9}}{1}$ the same as $\frac{20}{9}$?

If it is, then $\frac{20}{9} \div 1 = \frac{20}{9}$.

We do know that this is true since the product of 1 and a rational number is that same rational number.

Thus $\frac{5}{3} \div \frac{3}{4} = \frac{20}{9}$ and $\dfrac{\frac{5}{3}}{\frac{3}{4}} = \frac{20}{9}$; so, $\frac{5}{3} \div \frac{3}{4} = \dfrac{\frac{5}{3}}{\frac{3}{4}}$.

TEACHING MATERIALS

Physical models are of utmost importance for developing the concept of rational numbers, and of the product of two rational numbers. We suggest a number of kinds of materials which teachers have found effective. Such materials, and others which may be equally useful, should be used freely.

For Teacher:

(1) Cards A-W on pages 82, 83, and 84 should be copied by the teacher on foot square paper, preferably cardboard. Use a foot square of colored acetate to indicate shaded areas so that cards may be reused.

(2) Figures 1-12 on page 85 should be copied on cardboard by teacher.

(3) Number lines on page 86 should be copied on a large piece of paper. Number lines might be grouped or used individually. Suggested groups would be (1) A-E, (2) F-H, and (3) I-J.

(4) Fraction chart on page 87 should be enlarged for class use, on heavy paper.

(5) Models of Pocket Charts and cards for work in reading and writing decimals are suggested on page 88.

(6) Pictures of arrays which are used in this development to represent $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, and $\frac{1}{16}$ of a set of objects are shown on page 89 and could be enlarged on heavy paper for class use.

(7) Other devices

Flannel board and fractional parts
Ruler
Yardstick
Clock
Measuring cups showing halves, thirds, fourths, etc.
Scales

1" cubes

Play money

Pegboard

String marked with colored ink or colored beads to congruent line segments.

Strips of paper for folding

Circular, undecorated paper plates - one to represent the unit, others separated into congruent regions representing halves, thirds, fourths, fifths, sixths, eighths, and tenths.

Thermometer

Groups of concrete objects

Egg cartons

Carton dividers

Material for Pupils:

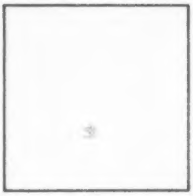
Dittoed copies of models used by teacher in development of unit, especially number lines and fraction chart.

Activities for Pupils:

Fraction "trees" suggested on page 91

Magic squares

Fraction Games - puzzle is suggested on page 90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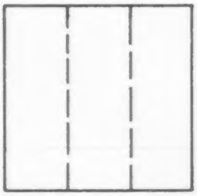
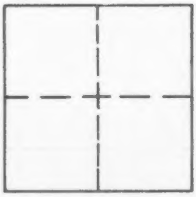
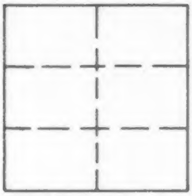
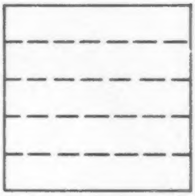
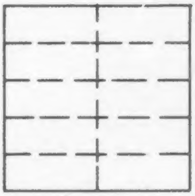

Card B


Card C


Card D


Card E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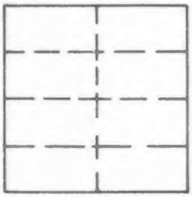
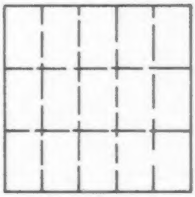

Card G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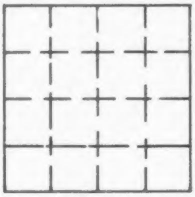
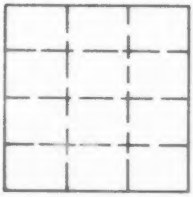
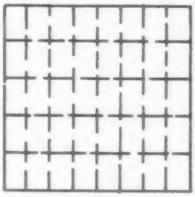
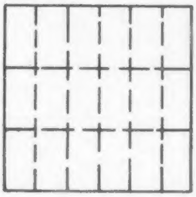
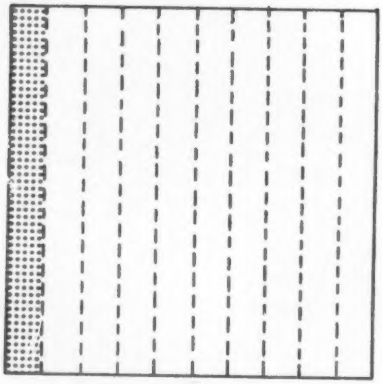

Card I

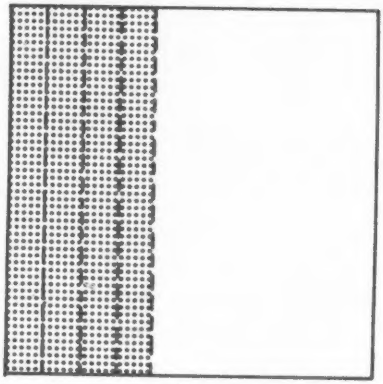

Card J


Card K

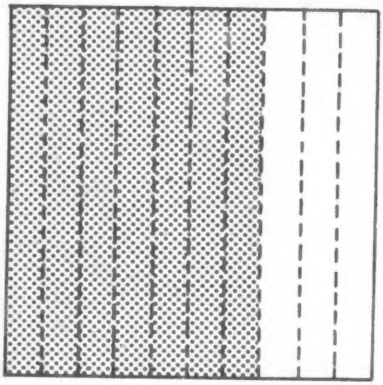

Card L



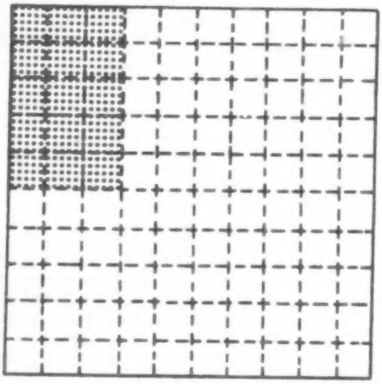
M



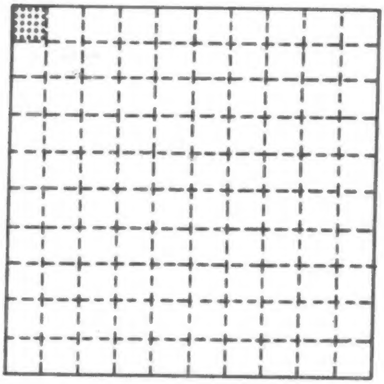
N



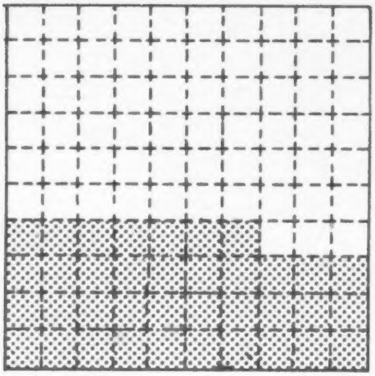
O



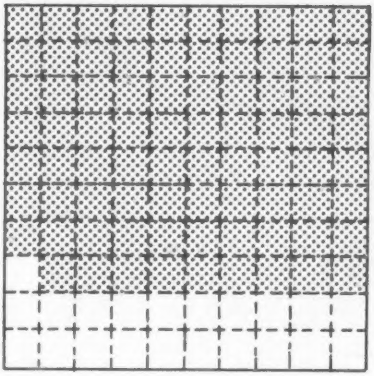
P



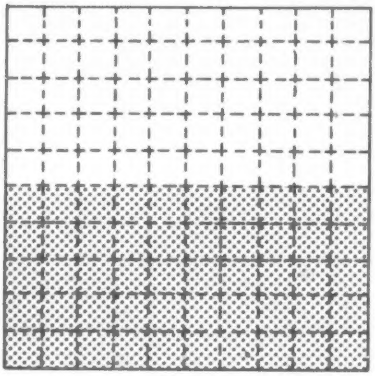
Q



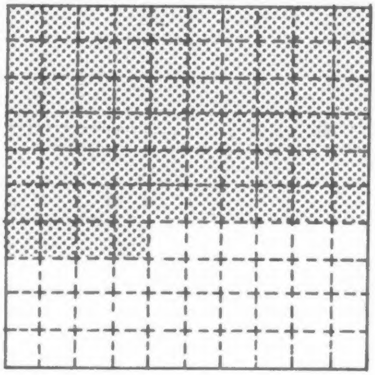
R



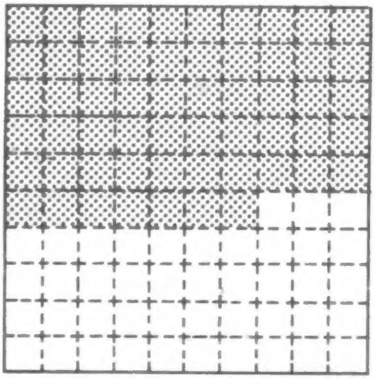
S



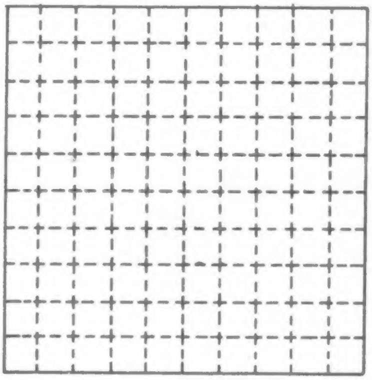
T



U



V



W

Models for Figures 1-12

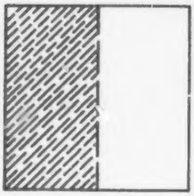


Figure 1

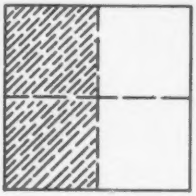


Figure 2

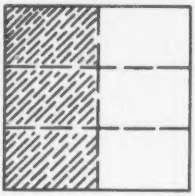


Figure 3

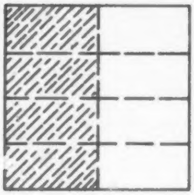


Figure 4

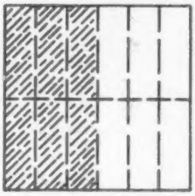


Figure 5

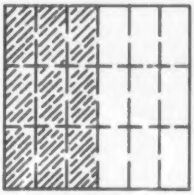


Figure 6

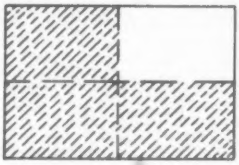


Figure 7

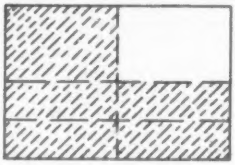


Figure 8

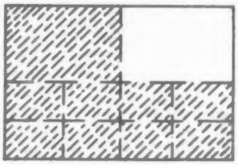


Figure 9

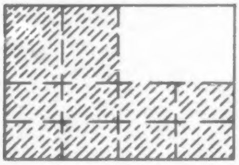


Figure 10

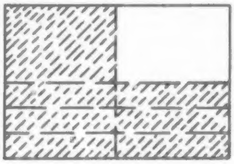


Figure 11

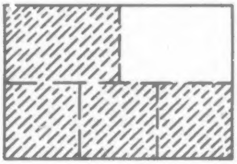
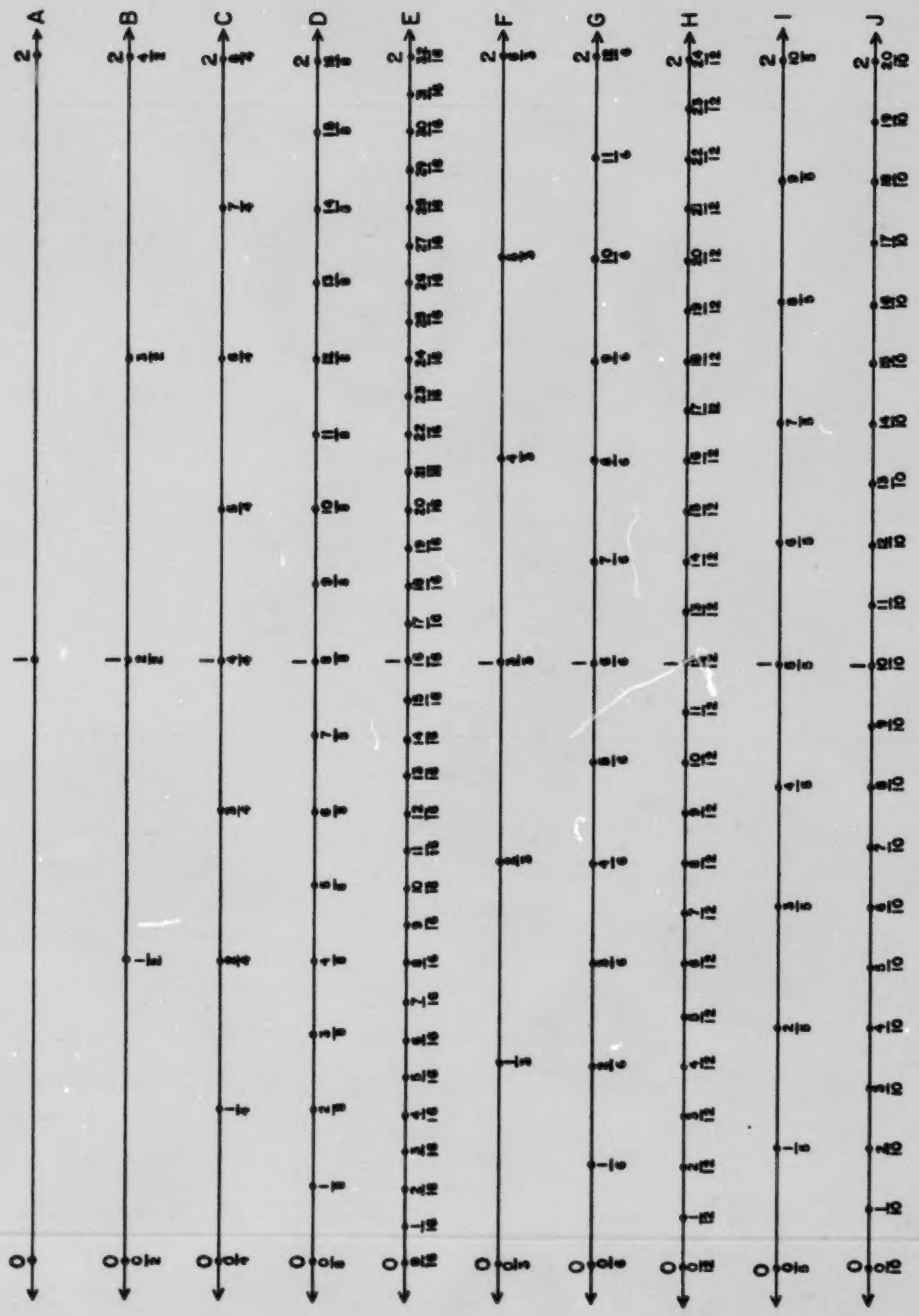
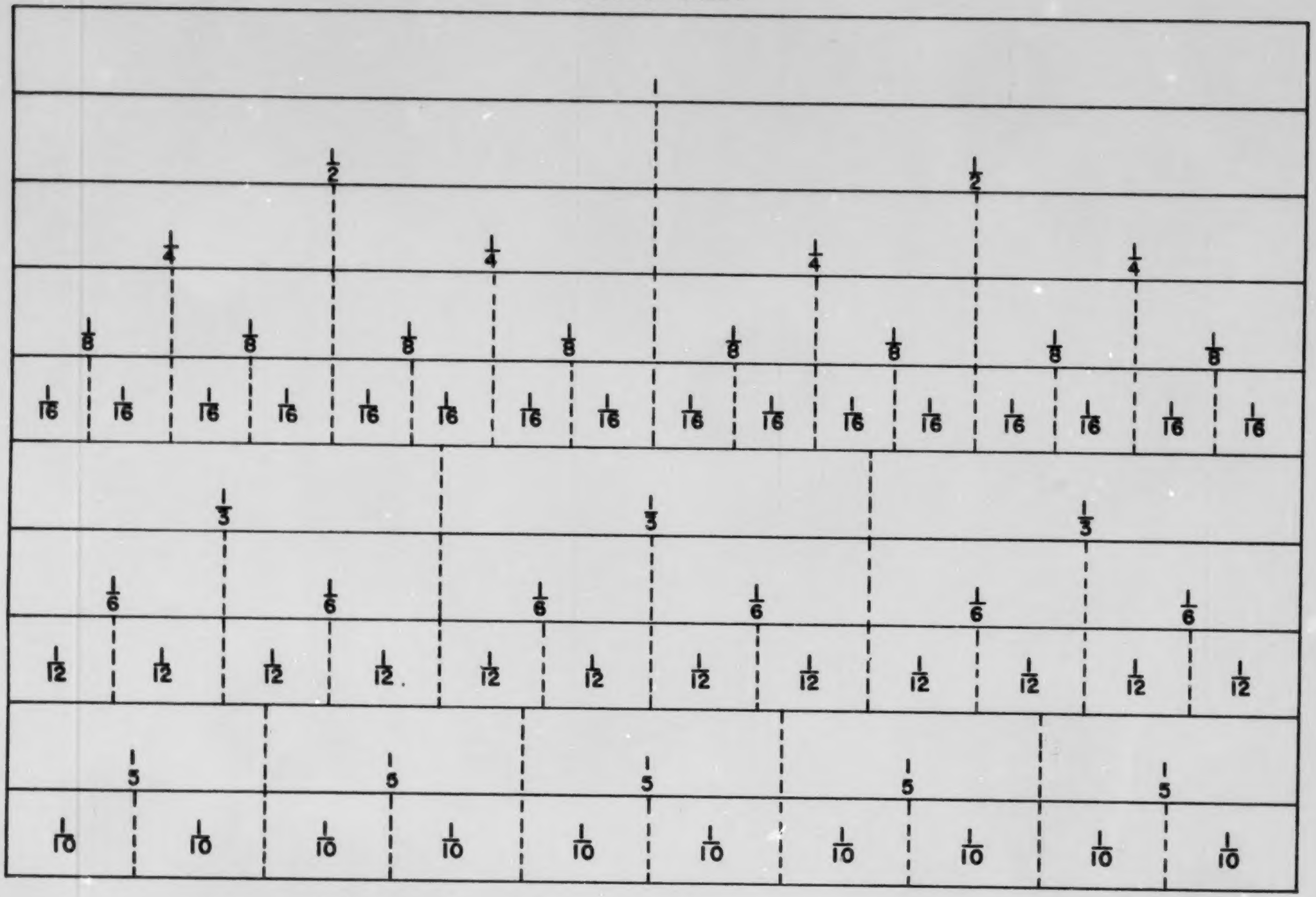


Figure 12


A
B
C
D
E
F
G
H
I
J
0
1
2

Fraction Chart


1/2
1/2
1/4
1/4
1/4
1/4
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/16
1/3
1/3
1/3
1/6
1/6
1/6
1/6
1/6
1/6
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/12
1/5
1/5
1/5
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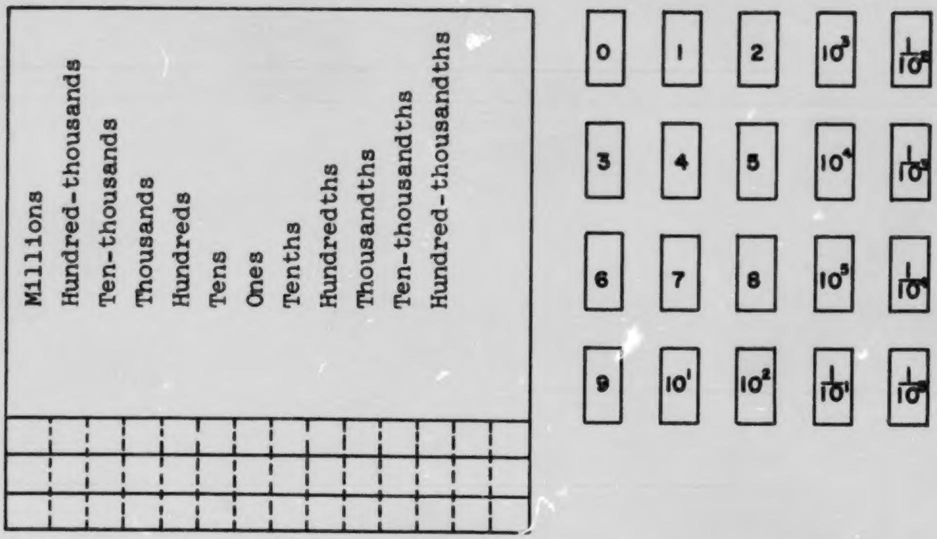
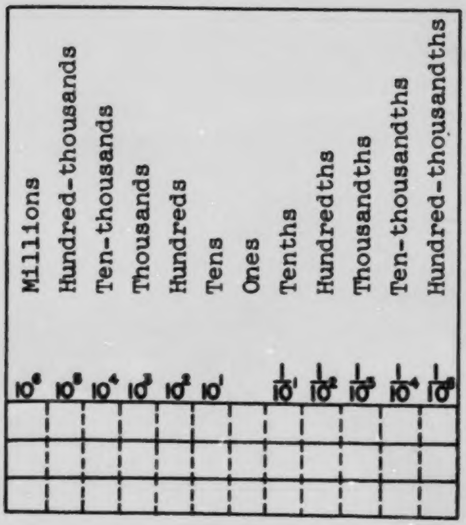
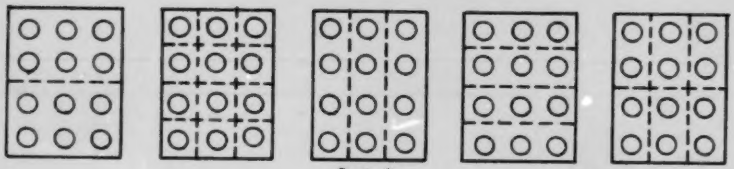
1/10
1/10
1/10
1/10


Millions
Hundred-thousands
Ten-thousands
Thousands
Hundreds
Tens
Ones
Tenths
Hundredths
Thousandths
Ten-thousandths
Hundred-thousandths
0
3
6
9
1
4
7
10^1
2
5
8
10^2
10^3
10^4
10^5
$\frac{1}{10^1}$
$\frac{1}{10^2}$
$\frac{1}{10^3}$
$\frac{1}{10^4}$
$\frac{1}{10^5}$

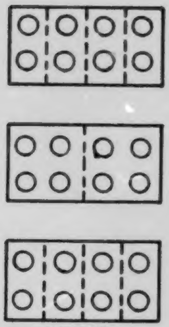
Pocket Chart and Cards for Place Value


10^6
10^5
10^4
10^3
10^2
10^1
$\frac{1}{10^1}$
$\frac{1}{10^2}$
$\frac{1}{10^3}$
$\frac{1}{10^4}$
$\frac{1}{10^5}$
Millions
Hundred-thousands
Ten-thousands
Thousands
Hundreds
Tens
Ones
Tenths
Hundredths
Thousandths
Ten-thousandths
Hundred-thousandths

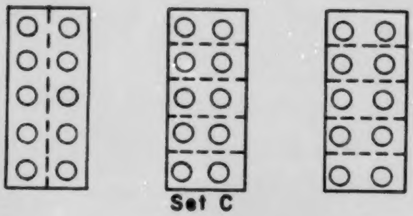
Arrays



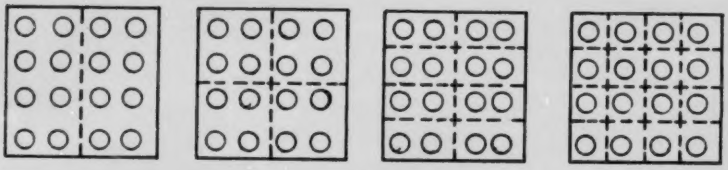
Set A



Set B



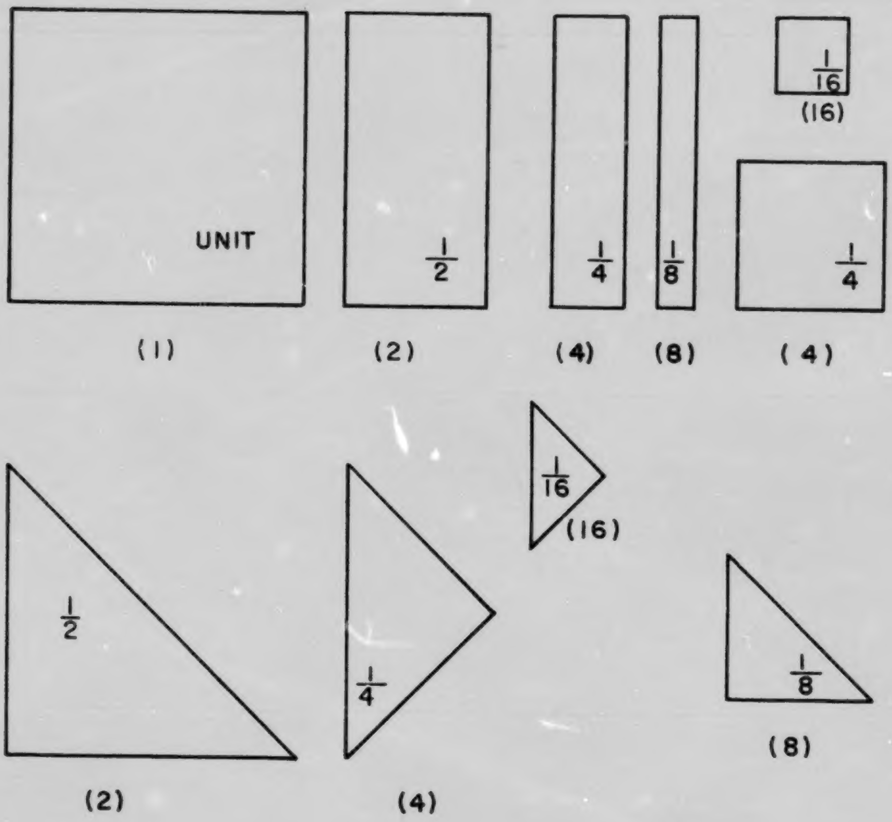
Set C



Set D

These arrays could be helpful in seeing $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, and $\frac{1}{16}$ of a set of objects. Used with acetate paper you could show $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, etc. of a set of objects.

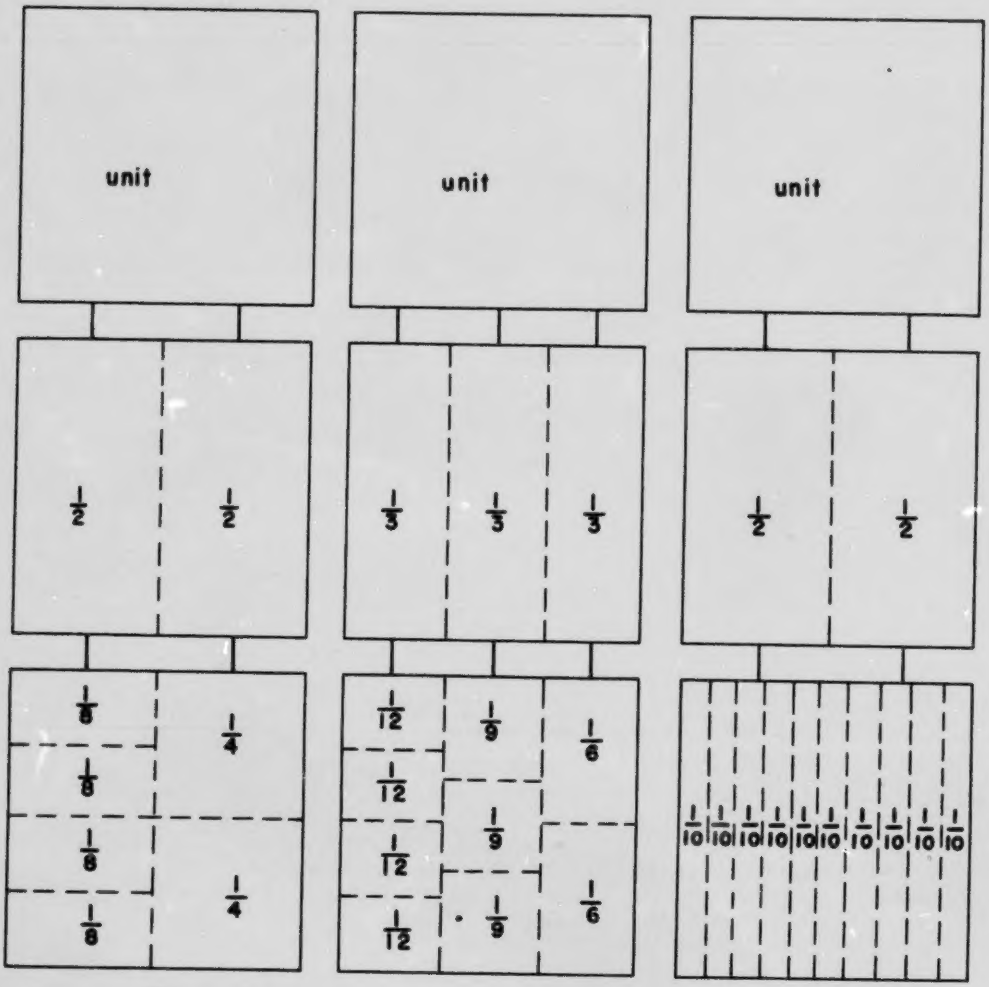
Puzzle


UNIT
(1)
1/2
(2)
1/4
(4)
1/8
(8)
1/16
(16)
1/4
(4)
1/2
(2)
1/4
(4)
1/16
(16)
1/8
(8)

Shallow top of square box may be used as frame. Cut out fractional parts shown above in proportion to frame. Numeral below each part indicates the number of that part to include. Let children see in how many different ways they can cover the unit.

You might also have puzzles for (1) halves, thirds, sixths, and twelfths and (2) halves, fifths, and tenths.

Fraction Trees


unit
unit
unit
1/2
1/2
1/3
1/3
1/3
1/2
1/2
1/8
1/8
1/8
1/8
1/4
1/4
1/12
1/12
1/12
1/12
1/9
1/9
1/9
1/6
1/6
1/10
1/10
1/10
1/10
1/10
1/10
1/10
1/10
1/10
1/10

TEACHING THE UNIT

RATIONAL NUMBER (REVIEW)

Objectives: To review the meaning of and the use of rational numbers.

To review the symbolism for rational numbers: fraction, decimal, and mixed form.

Materials: Cards A-G page 82 : colored acetate; number lines I, page 86 ; flannel board and models of unit regions cut into congruent regions.

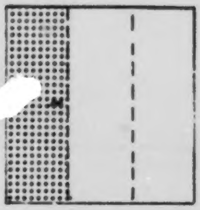
Vocabulary: Unit square region, unit line segment, congruent separated, measure, union, rational number, fraction, denominator, numerator, region, line segment.

Suggested Teaching Procedure:

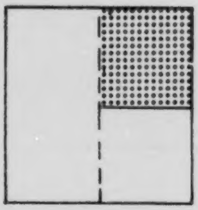
Teachers should use only as much of this section as is needed by the pupils in his class. It is intended to bring into focus those ideas associated with rational numbers which are used in the development of multiplication.

(a) Meaning of unit and part of unit.

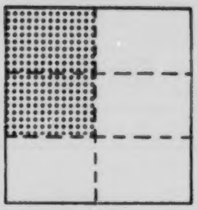
Have ready materials such as the Cards A through G on page 82 . The shading of the diagrams below shows how one might use transparent materials to cover that part of the region. Each card represents a unit region. The purpose of this development is to help identify that rational number which indicates the relationship between the part of the unit and the unit. Use also paper plates and felt pieces on flannel board to represent circular regions and number line I on page 86 .



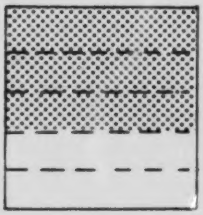
Card B



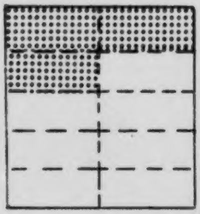
Card C



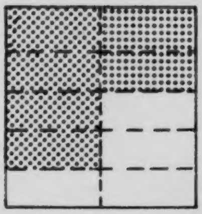
Card D



Card E



Card F



Card G

The class discussion can be developed in the following manner.

Look at these cards. (Models on page 93) They represent square regions. The dotted lines separate each region into smaller congruent regions. Look at Card B. If 1 is the measure of the region represented by the card, what number is the measure of the part that is shaded? ($\frac{1}{3}$) What part is unshaded? ($\frac{2}{3}$) What part of Card C is shaded? ($\frac{1}{4}$) What part is unshaded? ($\frac{3}{4}$)

Continue these questions with the other cards.

Why do you say $\frac{3}{10}$ of Card F is shaded? (The square region is separated into 10 congruent regions. Each of them is $\frac{1}{10}$ of the whole region. Three of these regions are shaded.)

$\frac{3}{10}$ is the measure of the shaded part of Card F.

The measure of the region represented by Card B is 1. The measure of the shaded region is $\frac{1}{3}$. What is the measure of the unshaded region? ($\frac{2}{3}$)

The measure of the region represented by Card D is 1. What is the measure of the unshaded region? ($\frac{4}{6}$) What is the measure of the shaded regions? ($\frac{2}{6}$)

Continue as needed.

Line segment. You may then wish to use paper plates or models on the flannel board to represent other regions such as circular regions, triangular regions, etc., before representing a line segment on the chalkboard. Mark the segment to show five congruent parts. Let the measure of the line segment be 1. Then ask for the measure of the union of two of the congruent segments; ($\frac{2}{5}$) the union of four of the congruent segments. ($\frac{4}{5}$) etc.

Observe other representations of rational numbers. You may wish to note that commonly we use a rational number to indicate the measure of a line segment or of a region.

Summarize these experiences by recording the measures of the shaded regions of the cards (or whatever materials were used). They might be; $\frac{1}{3}$; $\frac{1}{4}$; $\frac{2}{6}$; $\frac{3}{5}$; $\frac{3}{10}$; $\frac{5}{8}$.

(b) Vocabulary.

Each of these names is called a fraction. Each fraction names a particular pair of numbers. The number named below the line is called the denominator. Have pupils name the denominators in fractions listed. (3, 4, 6, etc.) Each of these fractions also has a numerator. It is the number named above the line. Name the numerators in the fractions listed. (1, 1, 2, etc.)

Continue to bring out the idea that a rational number may be illustrated by:

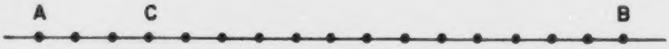
(1) Using regions.

Have the pupils look at the cards and tell what each denominator indicates. (The 3 of $\frac{1}{3}$ shows that the region has been separated into 3 smaller regions of equal measure; the 8 of $\frac{5}{8}$ shows that the region has been separated into 8 smaller regions of equal measure.)

Have them look at the cards and tell what each numerator indicates. (The 1 of $\frac{1}{3}$ tells that 1 of the 3 congruent regions is shaded on Card B; the 6 of the $\frac{6}{10}$ tells that 6 of the 10 congruent regions are shaded on Card G.

(2) Using line segments.

Represent a line segment AB on the chalkboard. State that it is to be the unit segment, so its measure is 1. Separate it intc 16 congruent parts. Locate point C so that $\overline{AC}$ is the union of 3 congruent parts. Ask for the measure of line segment $\overline{AC}$ ($\frac{3}{16}$), and then ask for the measure of line segment $\overline{CB}$ ($\frac{13}{16}$) and for the measure of line segment $\overline{AB}$ ($\frac{16}{16}$). Ask pupils how they determined the numerator and denominator of each fraction.



Identify other situations where rational numbers are used as:

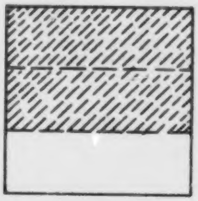
5 days is what part of a week
2 ounces is what part of a pound
7 inches is what part of a foot
4 apples is what part of 16 apples, etc.

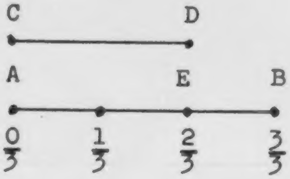
Chapter 2

RATIONAL NUMBERS

RATIONAL NUMBER (REVIEW)

Rational numbers may be used for the measure of a region and for the measure of a line segment. Think of other uses for rational numbers. Look at the figures below.





The shaded region is two-thirds of the square region. $\overline{AE}$ is two-thirds of $\overline{AB}$. Two-thirds is the measure of $\overline{CD}$.

Two-thirds may be represented by the numeral $\frac{2}{3}$, which is called a fraction. The 3 below the bar indicates that both the square region and $\overline{AB}$ have been separated into three parts of equal measure. The number 3 is called the denominator of the fraction.

The 2 above the bar in $\frac{2}{3}$ tells the number of thirds we are using. There are 2 thirds shaded in the square region and 2 thirds used for $\overline{AE}$. The number 2 is called the numerator of the fraction.

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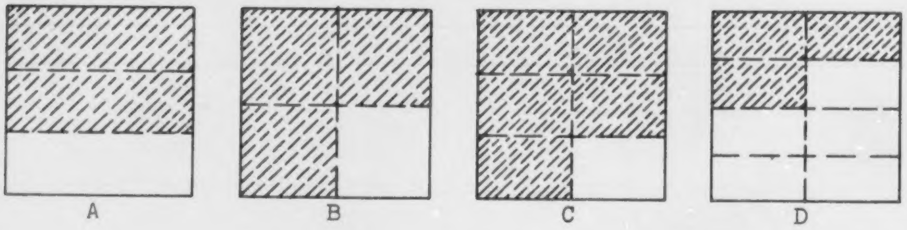
Exercise Set 1



1. Answer the following questions for each region above:

   a. Into how many parts is each separated? (A,3), (B,4), (C,6), (D,8)

   b. How many parts are shaded? (A - 2), (B - 3), (C - 5), (D - 3)

   c. If the measure of each large square region is 1, write the fraction that indicates the measure of the shaded region. (A - $\frac{2}{3}$), (B - $\frac{3}{4}$), (C - $\frac{5}{6}$), (D - $\frac{3}{8}$)

2. a. What do the denominators you wrote represent? (the number of congruent parts into which the large square region was separated)

   b. What do the numerators represent? (the number of congruent parts in the shaded region)

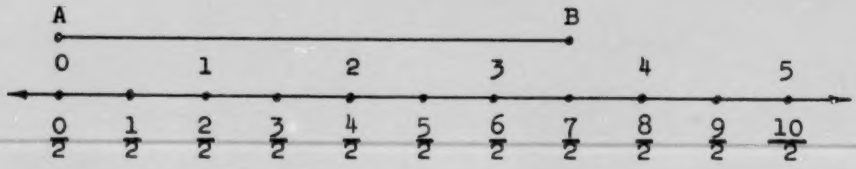
3. For each figure above, write the fraction that indicates the measure of the unshaded region. (A - $\frac{1}{3}$), (B - $\frac{1}{4}$), (C - $\frac{1}{6}$), (D - $\frac{5}{8}$)

4. a. What do the denominators you wrote for Exercise 3 represent? (the number of congruent parts into which the unit square region was separated)

   b. What do the numerators represent? (the number of congruent parts in the unshaded region)

   c. $\frac{3}{4} = \frac{2}{4} + \frac{1}{4}$ is shown in the shaded region of Figure B. Write similar mathematical sentences for the shaded parts of Figures A, C, and D. (A: $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$) (C: $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$, $\frac{1}{6} + \frac{4}{6} = \frac{5}{6}$), (D: $\frac{1}{8} + \frac{2}{8} = \frac{3}{8}$)

5.


A
B
0
1
2
3
4
5
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{3}{2}$
$\frac{4}{2}$
$\frac{5}{2}$
$\frac{6}{2}$
$\frac{7}{2}$
$\frac{8}{2}$
$\frac{9}{2}$
$\frac{10}{2}$

a. What fraction names the measure of $\overline{AB}$? ($\frac{7}{2}$)

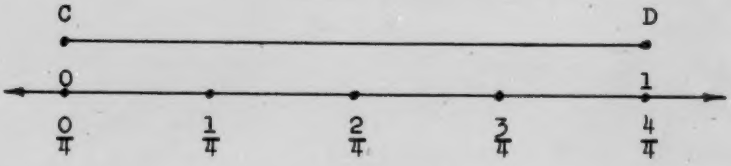
b. What does the denominator represent? (the number of congruent segments into which each unit of the number line is separated)

c. What does the numerator represent? (the number of congruent segments which line segment $\overline{AB}$ covers)

d. $\frac{2}{2} = \frac{1}{2} + \frac{1}{2}$. Write a mathematical sentence for your answer to Exercise 5a. Is more than one sentence possible? (yes)

($\frac{7}{2} = \frac{5}{2} + \frac{2}{2}$)
($\frac{7}{2} = \frac{1}{2} + \frac{6}{2}$)
($\frac{7}{2} = \frac{3}{2} + \frac{4}{2}$)

6.


C
D
0
1
$\frac{0}{4}$
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$

a. What fraction names the measure of line segment $\overline{CD}$? ($\frac{4}{4}$)

b. What does the denominator represent? (the number of congruent segments into which each unit is separated)

c. $\frac{2}{4} = \frac{1}{4} + \frac{1}{4}$. Write a mathematical sentence for your answer to Exercise 6a. ($\frac{3}{4} + \frac{1}{4} = \frac{4}{4}$), ($\frac{2}{4} + \frac{2}{4} = \frac{4}{4}$)

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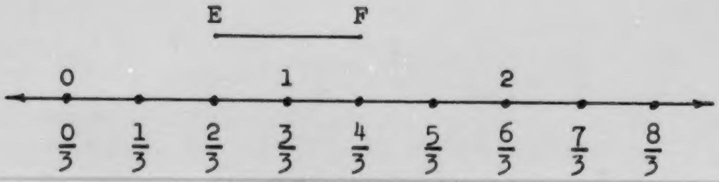
7.



E
F
0
1
2
$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
$\frac{7}{3}$
$\frac{8}{3}$

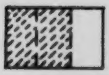
a. What is the measure of $\overline{EF}$? ($\frac{2}{3}$)

b. What does the denominator represent? (the number of congruent segments into which each unit is separated)

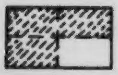
c. What does the numerator represent? (the number of congruent segments which $\overline{EF}$ covers)

d. Write a mathematical sentence about the measure of $\overline{EF}$.
($\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$)

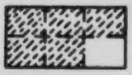
8. In Figures A - E, each figure represents a region whose measure is 1. Copy and complete the chart.



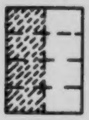
A



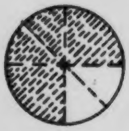
B



C



D



E

	Number of congruent Parts	Number of Parts Shaded	Measure of Shaded Region
A	3	2	$\frac{2}{3}$
B	(4)	(3)	($\frac{3}{4}$)
C	(6)	(5)	($\frac{5}{6}$)
D	(8)	(4)	($\frac{4}{8}$ or $\frac{1}{2}$)
E	(8)	(6)	($\frac{6}{8}$ or $\frac{3}{4}$)

9. Draw simple figures to show segments or regions whose measures are the numbers: $\frac{1}{4}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{7}{8}$. (Figures will vary)

10. a. What is the 2 called in these fractions: $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{11}$, $\frac{2}{25}$? (numerator)

b. What are the 3, 9, 11, and 25 called in the fractions? (denominator)

11. Complete the sentences below, using the Figures of Exercise 8.

a. The denominator of the fraction $\frac{3}{4}$ shows that Figure B in Exercise 8 has been separated into __(4)__ congruent parts.

b. The numerator of the fraction $\frac{6}{8}$ shows that Figure E has __(6)__ parts shaded.

c. The denominator of the fraction $\frac{2}{3}$ shows that Figure A has been separated into __(3)__ congruent parts.

d. The denominator of the fraction $\frac{4}{8}$ shows that Figure D has been separated into __(8)__ congruent parts.

Use rational numbers to answer these questions.

12. What part of a week is:

a. 1 day ($\frac{1}{7}$)

b. 3 days ($\frac{3}{7}$)

c. 5 days ($\frac{5}{7}$)

d. 7 days ($\frac{7}{7}$ or 1)

13. What part of one year is:

a. 9 months ($\frac{9}{12}$ or $\frac{3}{4}$)

b. 4 months ($\frac{4}{12}$ or $\frac{1}{3}$)

c. 6 months ($\frac{6}{12}$ or $\frac{1}{2}$)

d. 10 months ($\frac{10}{12}$ or $\frac{5}{6}$)

14. What part of an hour is:

a. 45 minutes ($\frac{45}{60}$ or $\frac{3}{4}$)

b. 30 minutes ($\frac{30}{60}$ or $\frac{1}{2}$)

c. 15 minutes ($\frac{15}{60}$ or $\frac{1}{4}$)

d. 10 minutes ($\frac{10}{60}$ or $\frac{1}{6}$)

15. What part of a pound is:

a. 4 ounces ($\frac{4}{16}$ or $\frac{1}{4}$)

b. 8 ounces ($\frac{8}{16}$ or $\frac{1}{2}$)

c. 12 ounces ($\frac{12}{16}$ or $\frac{3}{4}$)

d. 15 ounces ($\frac{15}{16}$)

16. What part of a yard is:

a. 1 foot ($\frac{1}{3}$)

b. 2 feet ($\frac{2}{3}$)

c. 3 feet ($\frac{3}{3}$ or 1)

d. 4 feet ($\frac{4}{3}$ or $1\frac{1}{3}$)

17. What part of a foot is:

a. 9 inches ($\frac{9}{12}$ or $\frac{3}{4}$)

b. 8 inches ($\frac{8}{12}$ or $\frac{2}{3}$)

c. 4 inches ($\frac{4}{12}$ or $\frac{1}{3}$)

d. 12 inches ($\frac{12}{12}$ or 1)

18. What part of a day is:

a. 6 hours ($\frac{6}{24}$ or $\frac{1}{4}$)

b. 60 minutes ($\frac{60}{1440}$ or $\frac{1}{24}$)

c. 8 hours ($\frac{8}{24}$ or $\frac{1}{3}$)

d. 12 hours ($\frac{12}{24}$ or $\frac{1}{2}$)

19. What part of a mile is:

a. 2,640 feet ($\frac{2640}{5280}$ or $\frac{1}{2}$)

b. 660 feet ($\frac{660}{5280}$ or $\frac{1}{8}$)

c. 1,320 feet ($\frac{1320}{5280}$ or $\frac{1}{4}$)

d. 330 feet ($\frac{330}{5280}$ or $\frac{1}{16}$)

DIFFERENT NAMES FOR A NUMBER (RATIONAL NUMBER)

Objectives: To review the idea that the same rational number can be named by different fractions, decimals, and mixed forms (e.g. $\frac{3}{2}$, $\frac{6}{4}$, $\frac{15}{10}$, 1.5, 1.50, $1\frac{1}{2}$, etc. are all numerals for the same rational number.)

To provide further practice in determining other names for a number when one or more names are already known.

Materials: Figures 1-6 page 103; dittoed copies, one for each pupil, are useful; each figure 1-6 represents a unit square region separated into smaller regions; number lines A, B, C, D, I, and J on page 86.

Suggested Teaching Procedure:

Different fractions for the same number.

The measure of the region represented by each of the six is 1. (See figures below) What is the measure of each of the shaded regions? (The shaded region of Figure 1 is $\frac{1}{2}$;...) Write these fractions on the chalkboard. ($\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{6}{12}$, $\frac{9}{18}$)

Ask if six unit square regions are all congruent. How can you tell? What do you notice about the shaded regions of all of them? (They are congruent.) Do they have the same measure? (Yes) To show in writing that these measures are the same, we write:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{6}{12} = \frac{9}{18}$$

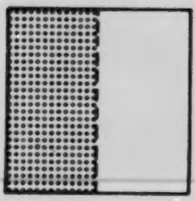


Fig. 1

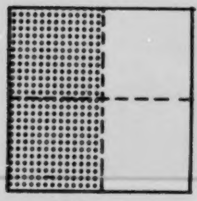


Fig. 2

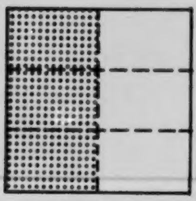


Fig. 3

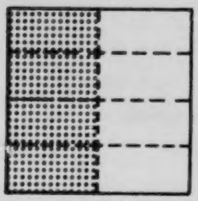


Fig. 4

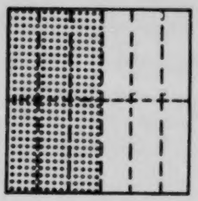


Fig. 5

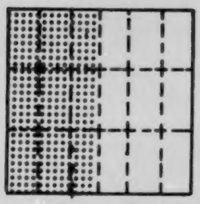


Fig. 6

Are there other fractions which name this same number? (Yes) List some. ($\frac{5}{10}$; $\frac{7}{14}$; etc.) How many could you list? (More than you can count.)

(Examine Fig. 1-6 again.) Which of them shows with dotted lines a region which is $\frac{1}{4}$ of the unit square region? (Fig. 2, 4, and 5). What is the measure of these as shown by the dotted lines of the paper? ($\frac{1}{4}$; $\frac{2}{8}$; $\frac{3}{12}$).

What is true about these fractions which you have used to name measures? (They name the same rational number.) Are there other ways of naming this number? (Yes. $\frac{4}{16}$; $\frac{20}{80}$; $\frac{100}{400}$). How many fractions belong to this set? (More than you can count.)

Write other names for each of these numbers. I will begin the sets of names for you.

Set A = $\{\frac{3}{4}, \frac{6}{8}, \qquad\qquad \}$

Set B = $\{\frac{8}{12}, \frac{2}{3}, \qquad\qquad \}$

Set C = $\{\frac{5}{8}, \frac{10}{16}, \qquad\qquad \}$

Set D = $\{\frac{2}{5}, \frac{10}{25}, \qquad\qquad \}$

How did you decide what other fractions to write in Set A? (I looked at $\frac{3}{4}$ of Figure 2 and folded it into smaller regions. Since each of the $\frac{1}{4}$ regions was separated into $\frac{2}{8}$, then $\frac{3}{4} = \frac{6}{8}$. This can be separated again......)

Let children mark models as needed to show other names to be included in Set A, Set B, etc.

Do you know another way to find a fraction that names a same rational number without using pictures? Examine the sets we have written on the chalkboard.

$$\text{Set A} = \{\tfrac{3}{4};\ \tfrac{6}{8};\ \tfrac{9}{12};\ \tfrac{12}{16};\ \tfrac{15}{20};\ \ldots;\ \tfrac{300}{400};\ \ldots\}$$

$$\text{Set B} = \{\tfrac{2}{3};\ \tfrac{4}{6};\ \tfrac{6}{9};\ \tfrac{8}{12};\ \ldots;\ \tfrac{16}{24};\ \ldots;\ \tfrac{20}{30};\ \ldots\}$$

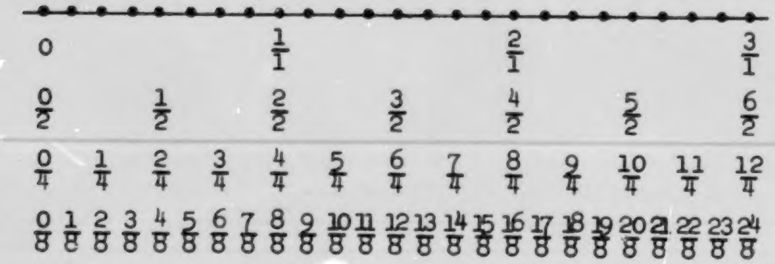
$$\text{Set C} = \{\tfrac{5}{8};\ \tfrac{10}{16};\ \tfrac{15}{24};\ \tfrac{20}{32};\ \tfrac{25}{40};\ \ldots;\ \tfrac{2500}{4000};\ \ldots\}$$

$$\text{Set D} = \{\tfrac{2}{5};\ \tfrac{4}{10};\ \ldots;\ \tfrac{10}{25};\ \ldots;\ \tfrac{200}{500};\ \ldots\}$$

Question the children about the relation of fractions in each of the sets. Help to generalize their observations as suggested below.

If you knew only the fraction $\frac{10}{16}$ in Set C and could not fold paper or make pictures, how could you find other names for the number $\frac{10}{16}$? (We could multiply both 10 and 16 by the same number.) Could this same plan be used for fractions from the other sets? (Yes) We could multiply the numerator and denominator of a fraction by the same number. Let's try this.

Teacher and children should follow a similar procedure until the children generalize from experience that the numerator and denominator of a fraction may be multiplied by the same number to find a fraction which names the same number. Then ask if any whole number can be used. Point out that the numerator and denominator are never multiplied by zero. Prepare number lines A-D on Page 86. Teachers should also construct a number line similar to the one below to show that different fractions name the same rational number.


0 1/1 2/1 3/1
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

DIFFERENT NAMES FOR A NUMBER

Regions are measured in terms of a unit square region. We may decide what unit square region we wish to use.

Let us use the unit square region shown in Figure 1. (We may sometimes call it a unit square instead of a unit square region. You must remember, however, we mean unit square region.)

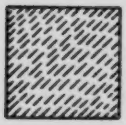


Figure 1

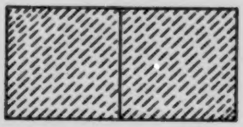


Figure 2

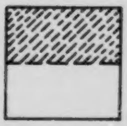


Figure 3

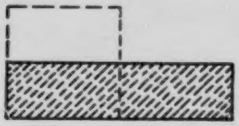
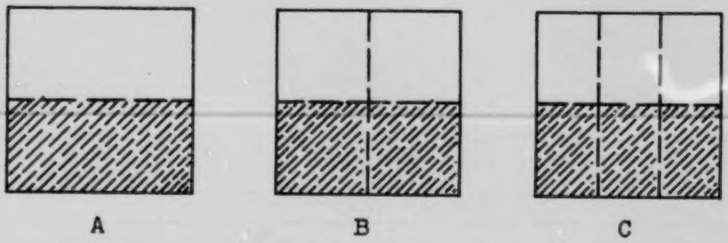


Figure 4

The shaded region of Figure 1 has the measure 1. The shaded region of Figure 2 has the measure 2 because it can be exactly covered by 2 unit squares. The shaded region of Figure 3 has the measure $\frac{1}{2}$ because it is one of the two congruent parts of a unit square.

A region may have the measure 1 and not have the same shape as the unit square of Figure 1. The shaded region of Figure 4 has measure 1. The unit square is sketched in with dotted lines to help you compare the shaded area with a unit square.



The unit square regions above are all the same size and shape. The measure of each is one.

Each region is separated into smaller congruent regions. One-half of Region A is shaded. Two-fourths of Region B is shaded. Three-sixths of Region C is shaded. The three shaded regions are the same size and shape. Their measures are equal.

The fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ are all names for the same rational number.

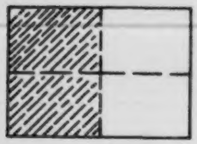
There are many other names for $\frac{1}{2}$. They may be found by drawing diagrams like A, B, and C. They may also be found by multiplying the numerator and denominator of the fraction by the same number.

A number has more names than you can count. Some other names for one-half are:

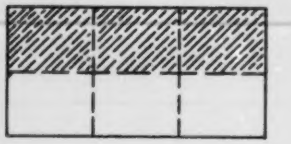
$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \cdots \frac{100}{200}, \cdots$$

Exercise Set 2

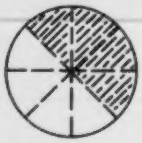
1. The shaded regions in A, B, and C suggest other names for $\frac{1}{2}$. Write these names. Write five other names for $\frac{1}{2}$. $(\frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}, etc.)$



A

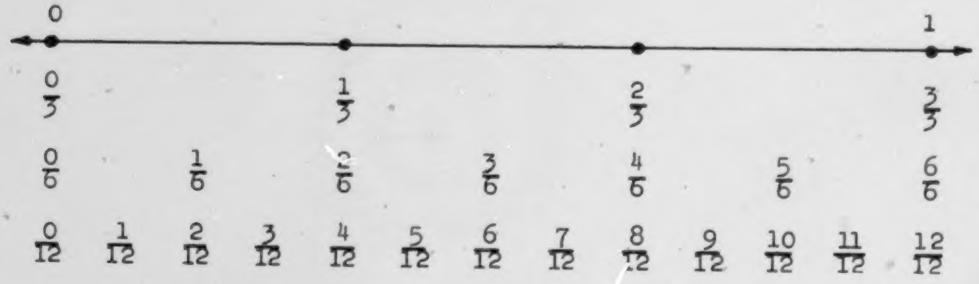


B



C

2. The number line below suggests different names for rational numbers $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{3}{3}$.



a. Write the names suggested on the number line for $\frac{2}{3}$. $(\frac{4}{6}, \frac{8}{12})$

b. Write three other names for $\frac{2}{3}$.

c. On the number line above, other names for 1 are suggested. Write these names, and three others.
$(\frac{3}{3}, \frac{6}{6}, \frac{12}{12})$, $(\frac{4}{4}, \frac{5}{5}, \frac{7}{7}, etc.)$

d. There is no other name for $\frac{1}{12}$ shown on the number line. Write one other name you know for $\frac{1}{12}$. $(\frac{2}{24})$

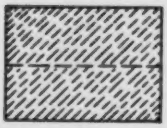
e. If the number line were extended to the point $\frac{4}{3}$, what other names would you write for $\frac{4}{3}$? $(\frac{8}{6}, \frac{16}{12})$

3. Draw simple figures to show

$$\frac{1}{4} = \frac{2}{8}; \quad \frac{1}{3} = \frac{2}{6}; \quad \frac{3}{4} = \frac{6}{8}; \quad 1 = \frac{2}{2}.$$

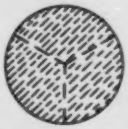
4. Write the fractions which rename 1 as suggested by the shad . regions in the figures below.

$(\frac{2}{2})$



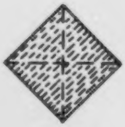
A

$(\frac{3}{3})$



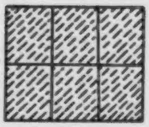
B

$(\frac{4}{4})$



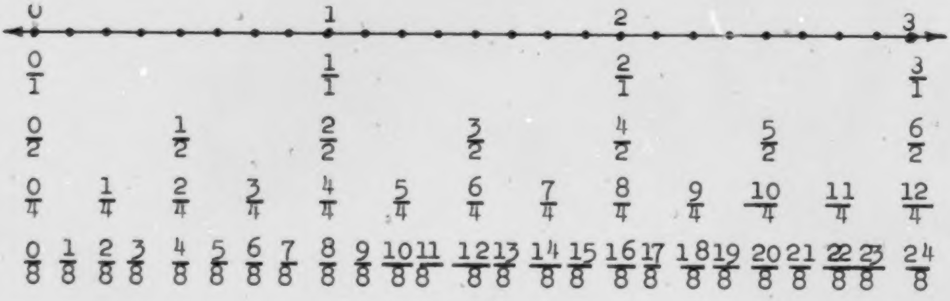
C

$(\frac{6}{6})$



D

5.


0 1 2 3
0/1 1/1 2/1 3/1
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

a. Write 3 other names for $1\frac{1}{2}$. $(\frac{3}{2}, \frac{6}{4}, \frac{12}{8})$

b. Write 2 other names for $\frac{16}{8}$. $(\frac{4}{2}, \frac{2}{1}, \frac{8}{4})$

c. Write four names for the number matching the point halfway between 2 and 3. $(\frac{5}{2}, \frac{10}{4}, \frac{20}{8}, \frac{40}{16}, 2\frac{1}{2})$

d. If the number line were extended to the point matching 4, write the set of fractions with denominator 4 that you would write below the number line. $(\frac{13}{4}, \frac{14}{4}, \frac{15}{4}, \frac{16}{4})$

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e. If you wrote $\frac{2}{3}$ on the number line in the row of fourths, between which two fractions would it be placed? $(\frac{3}{4}, \frac{2}{4})$

f. If you wrote a fraction for a number halfway between $\frac{6}{8}$ and $\frac{7}{8}$, what would it be? $(\frac{13}{16})$

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6. Below are sets of names for numbers.

Set A = $\{\frac{11}{22}, \frac{2}{4}, \frac{3}{6}, \frac{3}{4}, \frac{2}{1}, \frac{20}{10}, \frac{5}{10}, \frac{11}{21}\}$

Set B = $\{\frac{13}{39}, \frac{3}{7}, \frac{2}{6}, \frac{3}{9}, \frac{4}{10}, \frac{11}{31}, \frac{4}{12}, \frac{5}{15}\}$

a. What fractions in Set A are other names for $\frac{1}{2}$? $(\frac{11}{22}, \frac{2}{4}, \frac{3}{6}, \frac{5}{10})$

b. What fractions in Set B are other names for $\frac{1}{3}$? $(\frac{13}{39}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15})$

7. Which of the following rational numbers are the same as whole numbers?

$$\frac{12}{2}, \frac{17}{5}, \frac{3}{9}, \frac{71}{11}, \frac{50}{10}, \frac{9}{3}, 5\frac{1}{4}$$

$(\frac{12}{2} = 6)$, $(\frac{50}{10} = 5)$, $(\frac{9}{3} = 3)$

8. Write the names of whole numbers between $\frac{1}{2}$ and $\frac{9}{2}$. (1, 2, 3, 4)

9. Replace n with a numeral to make the statements below true.

a. $\frac{1}{2} = \frac{2}{n}$ (4)

b. $\frac{1}{4} = \frac{n}{8}$ (2)

c. $\frac{2}{4} = \frac{16}{n}$ (32)

d. $\frac{3}{4} = \frac{n}{28}$ (21)

e. $\frac{2}{3} = \frac{8}{n}$ (12)

f. $\frac{14}{16} = \frac{n}{8}$ (7)

g. $3 = \frac{9}{n}$ (3)

h. $1 = \frac{6}{n}$ (6)

i. $\frac{7}{4} = \frac{35}{n}$ (20)

j. $\frac{24}{8} = \frac{n}{3}$ (9)

10. Which number is greater?

a. $\frac{1}{2}$ or $\frac{1}{3}$ $(\frac{1}{2})$

b. $\frac{3}{4}$ or $\frac{2}{3}$ $(\frac{3}{4})$

c. $\frac{4}{4}$ or $\frac{1}{2}$ $(\frac{4}{4})$

d. $\frac{7}{3}$ or $\frac{2}{1}$ $(\frac{7}{3})$

e. $\frac{15}{16}$ or $\frac{3}{4}$ $(\frac{15}{16})$

f. $\frac{5}{8}$ or $\frac{1}{1}$ $(\frac{1}{1})$

11. Mark each statement true or false.

(T) a. $\frac{1}{2} = \frac{7}{14}$

(F) b. $\frac{1}{3} = \frac{1}{9}$

(T) c. $\frac{2}{3} = \frac{18}{27}$

(F) d. $\frac{1}{4} = \frac{4}{20}$

(T) e. $\frac{3}{4} = \frac{18}{24}$

(F) f. $\frac{2}{6} = \frac{8}{14}$

(T) g. $\frac{4}{6} = \frac{40}{60}$

(T) h. $\frac{2}{8} = \frac{22}{88}$

(F) i. $\frac{72}{12} = \frac{6}{6}$

(T) j. $\frac{88}{22} = 4$

(T) k. $\frac{24}{30} = \frac{4}{5}$

(F) l. $\frac{15}{3} = 18$

12. Complete, using ">", "<", or "=" in each blank. (Recall that ">" means "is greater than" and "<" means "is less than.")

a. $\frac{1}{2}$ (>) $\frac{1}{4}$

b. $\frac{3}{4}$ (<) $\frac{7}{8}$

c. 2 (>) $\frac{24}{16}$

d. $7\frac{1}{2}$ (=) $\frac{15}{2}$

e. $\frac{3}{4}$ (>) $\frac{8}{12}$

f. 5 (<) $\frac{60}{10}$

g. $3\frac{1}{3}$ (=) $\frac{10}{3}$

h. $\frac{2}{3}$ (<) $\frac{1}{1}$

13. Write fractions that make true statements.

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = (\frac{4}{8}) = (\frac{5}{10}) = (\frac{6}{12}) = (\frac{7}{14}) = (\frac{8}{16}) = (\frac{9}{18})$

$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = (\frac{4}{4}) = (\frac{5}{5}) = (\frac{6}{6}) = (\frac{7}{7}) = (\frac{8}{8}) = (\frac{9}{9})$

$\frac{24}{12} = \frac{12}{6} = (\frac{6}{3}) = (\frac{48}{24})$

$\frac{2}{3} = \frac{4}{6} = (\frac{6}{9}) = (\frac{8}{12}) = (\frac{10}{15}) = (\frac{12}{18}) = (\frac{14}{21})$

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14. Which of the following fractions are other names for $\frac{2}{3}$?

$\frac{6}{8}, \frac{6}{9}, \frac{8}{12}, \frac{9}{12}, \frac{5}{6}, \frac{10}{15}, \frac{9}{15}$ $(\frac{6}{9}, \frac{8}{12}, \frac{10}{15})$

15. Which of the following fractions are not names for $\frac{9}{12}$?

$\frac{2}{3}, \frac{3}{4}, \frac{24}{18}, \frac{27}{36}, \frac{6}{8}, \frac{18}{21}$ $(\frac{2}{3}, \frac{24}{18}, \frac{18}{21})$

16. Arrange in order from least to greatest.

$\frac{1}{2}, \frac{1}{3}, \frac{7}{4}, \frac{10}{1}, \frac{5}{6}, \frac{11}{2}$ $(\frac{1}{3}, \frac{1}{2}, \frac{5}{6}, \frac{7}{4}, \frac{11}{2}, \frac{10}{1})$

17. Name the greater number of each pair. See also 12a, 12b.

a. $\frac{5}{4}$ or $\frac{6}{3}$ $(\frac{6}{3})$

b. $\frac{2}{8}$ or $\frac{8}{2}$ $(\frac{8}{2})$

c. $\frac{1}{1}$ or $\frac{15}{16}$ $(\frac{1}{1})$

d. $\frac{17}{4}$ or $\frac{32}{5}$ $(\frac{32}{5})$

e. $\frac{8}{3}$ or $\frac{11}{4}$ $(\frac{11}{4})$

f. $\frac{5}{6}$ or $\frac{11}{12}$ $(\frac{11}{12})$

18. $\frac{3}{5}$ may be thought of as $\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$ or as "three $\frac{1}{5}$'s" or as $\frac{2}{5} + \frac{1}{5}$ or as $\frac{6}{10}$.

$4\frac{1}{2}$ may be thought of as $4 + \frac{1}{2}$ or $9 \times \frac{1}{2}$ or as

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

3.7 may be thought of as 3 + 0.7 or as 2.3 + 1.4.

Write three other names for each of these numbers:

a. $\frac{1}{2}$ b. $\frac{7}{4}$ c. 2.5

(Answers will vary.)

FRACTIONS AND MIXED FORMS

Objective: To review the idea that a rational number may be named by a mixed form as well as by a fraction; that is, $2\frac{1}{2}$ and $\frac{5}{2}$ both name the same number. (We call $2\frac{1}{2}$ a mixed form and we call $\frac{5}{2}$ a fraction.)

Vocabulary: Mixed form, simplest form, simplest mixed form

Suggested Teaching Procedure:

(a) Using physical models.

This is a review topic. You will find the number line or circular regions on the flannel board quite helpful. For example, locate point on number line that is associated with $\frac{9}{2}$. Also see that using unit segments, of 1, that you can also name this $4\frac{1}{2}$.

You may wish to consider the segment of the number line between 1 and 2. Note how you can use fractions and then mixed forms.

If circular regions are used, then show for example how $\frac{9}{4}$ can be arranged to form two unit regions and a $\frac{1}{4}$ unit.

(b) Using numerals.

Name rational numbers by fractions. Plan sequence of experiences so that pupils can observe that all rational numbers greater than one can be named using a numeral for a whole number and a fraction. (This is what we call a mixed form. You probably know it better by a mixed number.)

Fraction to Mixed Form.

Then, show what computation procedures may be used to go from a fraction name to a mixed form name.

$$\text{e.g.}\quad \frac{15}{4} = \frac{12}{4} + \frac{3}{4}$$

Show that we name the 15 as a sum of the largest multiple of 4, equal to or less than 15, plus that which is "left over."

Mixed Form to Fraction.

Show what computation procedures may be used to go from a mixed form name to a fraction name:

$$\text{e.g.}\quad 2\frac{3}{4} = \frac{2 \times 4}{1 \times 4} + \frac{3}{4} = \frac{8 + 3}{4} = \frac{11}{4}$$

It is often necessary or desirable to rename a number named by a fraction $\frac{a}{b}$, a greater than b, in mixed form. For example, $\frac{8}{6} = 1\frac{2}{6}$ or $1\frac{1}{3}$. The numeral $1\frac{1}{3}$ is called the simplest mixed form for this number.

The development of a method for renaming a number in mixed form is based on the mathematical sentence used for recording the result of division of whole numbers presented in Grade 5, Chapter 3. When 257 is divided by 5, the quotient is 51 and the remainder is 2. So $257 = (5 \times 51) + 2$. If your students have learned to divide and check by traditional methods and are not familiar with this way of recording the relation, they can be introduced to it by recalling the method they have learned.

$$\begin{array}{r} 51 \\ 5\overline{)257} \\ \underline{25} \\ 7 \\ \underline{5} \\ 2 \end{array} \qquad\qquad \text{Check:}\quad \begin{array}{r} 51 \\ \underline{\times 5} \\ 255 \\ \underline{+2} \\ 257 \end{array}$$

The division is correct if $257 = (5 \times 51) + 2$.

This type of mathematical sentence is described by the form $257 = (5 \times n) + r$. Since the remainder should be less than the divisor, it is required that r be less than 5.

FRACTIONS AND MIXED FORMS

It is easy to see that $2\frac{1}{4}$ and $\frac{9}{4}$ are names for the same number. $\frac{9}{4}$ is a fraction name and $2\frac{1}{4}$ is a mixed form for this number.

$$\begin{aligned} 2\tfrac{1}{4} &= 2 + \tfrac{1}{4} \\ &= \tfrac{2}{1} + \tfrac{1}{4} \\ &= \tfrac{2 \times 4}{1 \times 4} + \tfrac{1}{4} \\ &= \tfrac{8}{4} + \tfrac{1}{4} \\ 2\tfrac{1}{4} &= \tfrac{9}{4} \end{aligned}$$

$$\begin{aligned} \tfrac{9}{4} &= \tfrac{8}{4} + \tfrac{1}{4} \\ &= 2 + \tfrac{1}{4} \\ &= 2\tfrac{1}{4} \end{aligned}$$

Suppose you have less familiar numbers.

$$\begin{aligned} 18\tfrac{5}{13} &= \tfrac{18}{1} + \tfrac{5}{13} \\ &= \tfrac{18 \times 13}{1 \times 13} + \tfrac{5}{13} \\ &= \tfrac{234}{13} + \tfrac{5}{13} \\ 18\tfrac{5}{13} &= \tfrac{239}{13} \end{aligned}$$

Consider the number $\frac{174}{8}$. How can it be written in mixed form? You know that $\frac{8}{8} = 1$.

$$\begin{aligned} \tfrac{174}{8} &= \tfrac{8 + 166}{8} \\ &= \tfrac{8}{8} + \tfrac{166}{8} \\ &= 1\tfrac{166}{8} \end{aligned}$$

or

$$\begin{aligned} \tfrac{174}{8} &= \tfrac{16 + 158}{8} \\ &= \tfrac{16}{8} + \tfrac{158}{8} \\ &= 2 + \tfrac{158}{8} \\ &= 2\tfrac{158}{8} \end{aligned}$$

Recall that a fraction is in simplest form when the numerator and denominator have no common factor, except 1. In the simplest mixed form for a rational number, the fraction is in simplest form and names a number less than 1. Is either mixed form above in simplest mixed form? To find the simplest mixed form for $\frac{174}{8}$ write the numerator, 174, in the form $(8 \times n) + r$, with r less than 8. What operation do you use to find n and r ?

$$174 = (8 \times n) + r$$

$$174 = (8 \times 21) + 6$$

$$\frac{174}{8} = \frac{(8 \times 21) + 6}{8}$$

$$= \frac{8 \times 21}{8} + \frac{6}{8}$$

$$\frac{174}{8} = 21 + \frac{6}{8} = 21\frac{6}{8} = 21\frac{3}{4}$$

Exercise Set 3

Write names for these numbers in the form shown. In Exercise 1, be sure $r < 6$.

1. $38 = (6 \times n) + r$ $(n=6, r=2)$
2. $55 = (3 \times n) + r$ $(n=18, r=1)$
3. $72 = (12 \times n) + r$ $(n=6, r=0)$
4. $69 = (14 \times n) + r$ $(n=4, r=13)$
5. $124 = (25 \times n) + r$ $(n=4, r=24)$
6. $347 = (18 \times n) + r$ $(n=19, r=5)$

Name these numbers in simplest mixed form. Show your work as in the examples above Exercise Set 3.

7. $\frac{50}{8}$ $\left(6\frac{1}{4}\right)$
8. $\frac{71}{7}$ $\left(10\frac{1}{7}\right)$
9. $\frac{41}{9}$ $\left(4\frac{5}{9}\right)$
10. $\frac{87}{10}$ $\left(8\frac{7}{10}\right)$
11. $\frac{145}{15}$ $\left(9\frac{2}{3}\right)$
12. $\frac{296}{18}$ $\left(16\frac{4}{9}\right)$

Find fraction names for these numbers.

13. $8\frac{3}{8}$ $\left(\frac{67}{8}\right)$
14. $9\frac{5}{8}$ $\left(\frac{77}{8}\right)$
15. $12\frac{3}{5}$ $\left(\frac{63}{5}\right)$
16. $23\frac{9}{11}$ $\left(\frac{262}{11}\right)$
17. $50\frac{3}{4}$ $\left(\frac{203}{4}\right)$
18. $29\frac{5}{7}$ $\left(\frac{208}{7}\right)$

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

Objective:	To review addition and subtraction of rational numbers using fractions, decimals and mixed forms.
Materials:	Have Fig. 7 through 12 page 85 available for use.
Vocabulary:	Prime factorization, greatest common factor.

Suggested Teaching Procedure:

(a) Describing the union of regions by addition.

Look at these figures which are separated into congruent parts. What part of each figure is shaded? ($\frac{3}{4}$ of each is shaded.) Some of the parts which show $\frac{3}{4}$ have been separated into smaller parts.

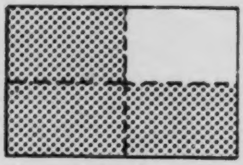


Fig. 7

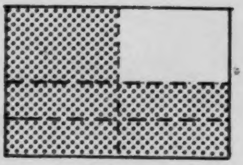


Fig. 8

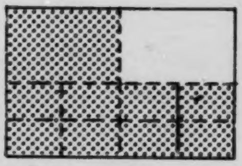


Fig. 9

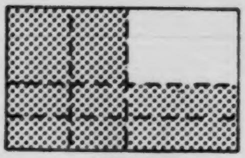


Fig. 10

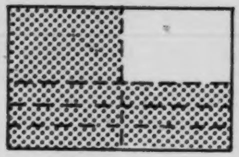


Fig. 11

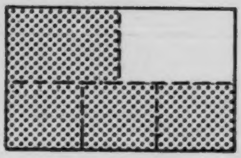


Fig. 12

Have children describe the shaded part of Figure 8 as $\frac{3}{4} = \frac{1}{4} + \frac{4}{8}$, the shaded part of Figure 9 as $\frac{3}{4} = \frac{1}{4} + \frac{8}{16}$, etc.

(b) Adding rational numbers.

We can use these figures to find the sum of two rational numbers. Find these sums:

(a) $\frac{2}{4} + \frac{1}{4} = n$ (c) $\frac{1}{12} + \frac{5}{12} = n$

(b) $\frac{3}{16} + \frac{3}{16} = n$ (d) $\frac{1}{6} + \frac{2}{6} = n$

(Figure 7 shows that $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. Figure 10 shows that $\frac{3}{16} + \frac{5}{16} = \frac{8}{16}$. Figure 11 shows that $\frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$. Figure 12 shows that $\frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$.)

Did you need these figures to find the sum of these numbers? (No. They were named by fractions with the same denominators.) What procedure do you use to add two numbers with the same denominator? (To find the numerator of the fraction name of the sum add the numerators of the fractions. The denominator of the fraction name of the sum is the denominator of the fractions.)

Some numbers which we wish to add are not named by fractions with the same denominator. Look at these.

> The numbers in the exercises below are those whose sums can be read from the figures. If children recognize that $\frac{8}{16} = \frac{2}{4}$ or $\frac{1}{2}$ they should not be discouraged by the teacher. At the same time the teacher may point out that these numbers are being used to find a method for the addition of any rational numbers.

(e) $\frac{1}{4} + \frac{8}{16} = n$ (f) $\frac{1}{4} + \frac{3}{6} = n$ (g) $\frac{1}{2} + \frac{1}{4} = n$

After expressing this sum of two numbers, children may find it easier to use vertical form to compute sum.

To add two numbers named by fractions with different denominators, you can first name them by fractions with the same denominator. How can you rename $\frac{1}{4}$ and $\frac{8}{16}$ in Exercise (e) so they have the same denominator? (Rename $\frac{1}{4}$ as $\frac{4}{16}$.) What is the sum? ($\frac{4}{16} + \frac{8}{16} = \frac{12}{16}$)

> Repeat questions for Exercise (f) and (g).

The renaming and addition is sometimes written like this:

$$\text{(e)}\ \begin{array}{r} \frac{1}{4} = \frac{4}{16} \\ \frac{8}{16} = \frac{8}{16} \\ \hline \frac{12}{16} \end{array} \qquad \text{(f)}\ \begin{array}{r} \frac{1}{4} = \frac{3}{12} \\ \frac{3}{6} = \frac{6}{12} \\ \hline \frac{9}{12} \end{array} \qquad \text{(g)}\ \begin{array}{r} \frac{1}{2} = \frac{2}{4} \\ \frac{1}{4} = \frac{1}{4} \\ \hline \frac{3}{4} \end{array}$$

Write only what you need. Some of you may not even need to write $\frac{4}{16}$. You may just think it.

In Exercise (e) and (f) the sums should be expressed in simpler form. Give the simpler names. ($\frac{12}{16} = \frac{3}{4}$; $\frac{9}{12} = \frac{3}{4}$)

> These sums may also be found by use of Figures 9, 12, and 7. The children may need further review.

(c) Subtracting rational numbers.

Looking at Figures 7 through 12 again, can you find the unknown addend in these subtraction examples:

(h) $\frac{3}{4} = \frac{1}{4} + n$ (i) $\frac{5}{8} = \frac{3}{8} + n$ (j) $\frac{5}{12} = \frac{1}{12} + n$

(From Fig. 7, $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$. From Fig. 8, $\frac{5}{8} - \frac{3}{8} = \frac{2}{8}$. From Fig. 11, $\frac{5}{12} - \frac{1}{12} = \frac{4}{12}$).

Do you need to use the figures to find the unknown addend of these numbers? (No. It is easy to subtract them because the fractions have the same denominator. We subtract the numerators.) What is the denominator of the fraction name of the unknown numbers which are subtracted.)

How are these numbers subtracted?

(k) $\frac{5}{8} = \frac{1}{4} + n$ (l) $\frac{3}{4} = \frac{5}{8} + n$ (m) $\frac{3}{4} = \frac{1}{6} + n$

(They can be renamed so the fractions have the same denominator.) How will you rename $\frac{5}{8}$ and $\frac{1}{4}$ to subtract them? (Rename $\frac{1}{4}$ as $\frac{2}{8}$. Then $\frac{5}{8} - \frac{2}{8} = \frac{3}{8}$).

How will you subtract in Exercise (l)? (Rename $\frac{3}{4}$ as $\frac{6}{8}$. $\frac{6}{8} - \frac{5}{8} = \frac{1}{8}$)

How will you subtract in Exercise (m)? (Rename $\frac{3}{4}$ as $\frac{9}{12}$ and $\frac{1}{6}$ as $\frac{2}{12}$. $\frac{9}{12} - \frac{2}{12} = \frac{7}{12}$.)

The form of recording your renaming and the subtraction is sometimes written like this:

(k) $$\begin{array}{r} \frac{5}{8} = \frac{5}{8} \\ \underline{\frac{1}{4} = \frac{2}{8}} \\ \frac{3}{8} \end{array}$$

(l) $$\begin{array}{r} \frac{3}{4} = \frac{6}{8} \\ \underline{\frac{5}{8} = \frac{5}{8}} \\ \frac{1}{8} \end{array}$$

(m) $$\begin{array}{r} \frac{3}{4} = \frac{9}{12} \\ \underline{\frac{1}{6} = \frac{2}{12}} \\ \frac{7}{12} \end{array}$$

(d) Summary.

The children may need further review.

Have the children summarize the method of adding and subtracting rational numbers:

(a) to add rational numbers name them by fractions that have the same denominator; add the numerators to find the numerator of the fraction name of the sum; the denominator of the sum is the denominator of the fractions for the numbers added.

(b) to subtract rational numbers name them by fractions of the same denominator; subtract the numerators to find the numerator of the fraction name of the unknown addend; the denominator of the unknown addend is the denominator of the fractions for the numbers subtracted.

The review includes the addition and subtraction of numbers written in mixed form, e.g., $4\frac{1}{2} + 2\frac{1}{3}$ and $4\frac{1}{2} - 2\frac{1}{3}$. It may be necessary to use class time for review of these processes.

ADDING AND SUBTRACTING RATIONAL NUMBERS

Recall that it is easy to add or subtract two rational numbers if they are named by fractions with the same denominator.

$$\frac{2}{5} + \frac{1}{5} = \frac{2 + 1}{5} = \frac{3}{5}$$

$$\frac{4}{3} - \frac{2}{3} = \frac{4 - 2}{3} = \frac{2}{3}$$

$$1\frac{3}{8} - \frac{5}{8} = \frac{11 - 5}{8} = \frac{6}{8} = \frac{3}{4}$$

To add or subtract rational numbers named by fractions with the same denominator, we add or subtract the numerators to find the numerator of the result. The denominator of the result is the same as the denominator of the two original fractions.

If the denominators of the two fraction *names* are not the same, one or both rational numbers are renamed so the fractions have the same denominator.

Add:

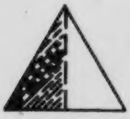
$$\begin{array}{ll} \frac{2}{3} = \frac{10}{15} & \qquad\qquad 2\frac{1}{2} = 2\frac{3}{6} \\ \frac{3}{5} = \underline{\frac{9}{15}} & \qquad\qquad 3\frac{1}{3} = \underline{3\frac{2}{6}} \\ \quad\;\; \frac{19}{15} = 1\frac{4}{15} & \qquad\qquad \quad\;\; 5\frac{5}{6} \end{array}$$

Subtract:

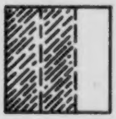
$$\begin{array}{ll} \frac{5}{8} = \frac{5}{8} & \qquad\qquad 2\frac{3}{4} = 2\frac{6}{8} = 1\frac{14}{8} \\ \frac{1}{2} = \underline{\frac{4}{8}} & \qquad\qquad 1\frac{7}{8} = 1\frac{7}{8} = \underline{1\frac{7}{8}} \\ \quad\;\; \frac{1}{8} & \qquad\qquad \qquad\qquad\quad \frac{7}{8} \end{array}$$

Exercise Set 4

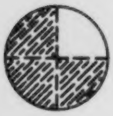
1.



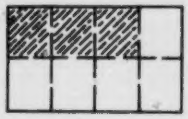
A



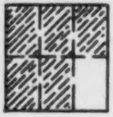
B



C



D



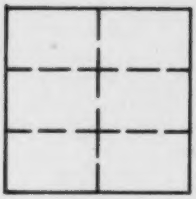
F

a. Figure A pictures the addition fact $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$. What addition facts are suggested by the shaded and unshaded regions of Figures B? C? D? and E?
$(B - \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1)$ $(C - \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1)$ $(D - \frac{3}{8} + \frac{5}{8} = \frac{8}{8} = 1)$ $(E - \frac{5}{6} + \frac{1}{6} = \frac{6}{6} = 1)$

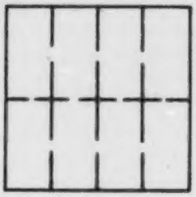
b. Figure B also shows that $1 - \frac{2}{3} = \frac{1}{3}$ and $1 - \frac{1}{3} = \frac{2}{3}$. What subtraction facts are suggested by the shaded and unshaded regions of Figure A? Figure C? Figure D? Figure E? $(A - 1 - \frac{1}{2} = \frac{1}{2})$ $(C - 1 - \frac{3}{4} = \frac{1}{4})$ $(D - 1 - \frac{3}{8} = \frac{5}{8})$
$(E - 1 - \frac{5}{6} = \frac{1}{6})$

P52

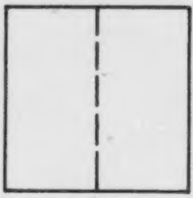
2.



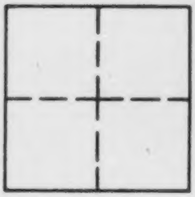
A



B



C



D

Choose one of the rectangular regions A, B, C, or D to find the sums of the numbers in the chart below.

	Numbers	Rectangular Region Used	Sum
a.	$\frac{1}{2}$ and $\frac{3}{8}$	B	$\frac{7}{8}$
b.	$\frac{1}{4}$ and $\frac{5}{8}$	(B)	($\frac{7}{8}$)
c.	$\frac{1}{2}$ and $\frac{1}{3}$	(A)	($\frac{5}{6}$)
d.	$\frac{1}{2}$ and $\frac{1}{4}$	(D)	($\frac{3}{4}$)

3. Recall what is meant by the "prime factorization" of a counting number. Complete:

	Fraction	Prime Factorization of Denominator	Prime Factorization of Numerator	Greatest Common Factor	Simplest Form
a.	$\frac{6}{8}$	$2 \times 2 \times 2$	2×3	2	$\frac{3}{4}$
b.	$\frac{8}{10}$	2×5	$(2 \times 2 \times 2)$	(2)	$(\frac{4}{5})$
c.	$\frac{9}{12}$	$(2 \times 2 \times 3)$	(3×3)	(3)	$(\frac{3}{4})$
d.	$\frac{2}{8}$	$(2 \times 2 \times 2)$	(2×1)	(2)	$(\frac{1}{4})$
e.	$\frac{8}{12}$	$(2 \times 2 \times 3)$	$(2 \times 2 \times 2)$	(4)	$(\frac{2}{3})$
f.	$\frac{14}{16}$	$(2 \times 2 \times 2 \times 2)$	(2×7)	(2)	$(\frac{7}{8})$
g.	$\frac{18}{6}$	(2×3)	$(2 \times 3 \times 3)$	(6)	$(\frac{3}{1})$
h.	$\frac{9}{6}$	(2×3)	(3×3)	(3)	$(1\frac{1}{2}$ or $\frac{3}{2})$
i.	$\frac{6}{9}$	(3×3)	(2×3)	(3)	$(\frac{2}{3})$
j.	$\frac{54}{27}$	$(3 \times 3 \times 3)$	$(3 \times 3 \times 3 \times 2)$	(27)	(2)

4. Rename each pair of numbers below so the fractions have the same denominator. Use the smallest denominator possible.

a. $\frac{1}{2}$ and $\frac{2}{3}$ $(\frac{3}{6}, \frac{4}{6})$ c. $\frac{2}{3}$ and $\frac{3}{4}$ $(\frac{8}{12}, \frac{9}{12})$ e. $\frac{3}{4}$ and $\frac{4}{5}$ $(\frac{15}{20}, \frac{16}{20})$

b. $\frac{5}{8}$ and $\frac{3}{4}$ $(\frac{5}{8}, \frac{6}{8})$ d. $\frac{1}{2}$ and $\frac{7}{10}$ $(\frac{5}{10}, \frac{7}{10})$ f. $\frac{2}{3}$ and $\frac{3}{5}$ $(\frac{10}{15}, \frac{9}{15})$

5. **Express in simplest form:**

a. $\frac{6}{8}\left(\frac{3}{4}\right)$ d. $\frac{8}{12}\left(\frac{2}{3}\right)$ g. $\frac{12}{8}\left(1\frac{1}{2}\right)$

b. $\frac{14}{10}\left(1\frac{2}{5}\right)$ e. $\frac{10}{15}\left(\frac{2}{3}\right)$ h. $1\frac{4}{3}\left(2\frac{1}{3}\right)$

c. $\frac{15}{3}\left(\frac{5}{1}\right)$ f. $\frac{7}{7}\left(1\right)$ i. $\frac{11}{4}\left(2\frac{3}{4}\right)$

6. **Find the sum of each pair of numbers and express it in its simplest form:**

a. $\frac{7}{8}$ and $\frac{3}{4}\left(1\frac{5}{8}\right)$ f. $\frac{3}{4}$ and $\frac{4}{5}\left(1\frac{11}{20}\right)$

b. $\frac{5}{12}$ and $\frac{2}{3}\left(1\frac{1}{12}\right)$ g. $\frac{9}{8}$ and $6\left(7\frac{1}{8}\right)$

c. $2\frac{1}{2}$ and $3\left(5\frac{1}{2}\right)$ h. $7\frac{1}{2}$ and $2\frac{4}{5}\left(10\frac{3}{10}\right)$

d. $13\frac{2}{3}$ and $34\frac{3}{4}\left(48\frac{5}{12}\right)$ i. $\frac{2}{3}$ and $\frac{7}{9}\left(1\frac{4}{9}\right)$

e. $12\frac{1}{2}$ and $4\frac{2}{5}\left(16\frac{9}{10}\right)$ j. $4\frac{1}{2}$ and $5\frac{2}{3}\left(10\frac{1}{6}\right)$

7. **Find n. Express n in its simplest form.**

a. $n = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{4}\right)$ e. $n = 1\frac{7}{12} - \frac{3}{4}\left(\frac{5}{6}\right)$ h. $n = 9\frac{2}{3} - 3\frac{3}{4}$ $\left(5\frac{11}{12}\right)$

b. $n = \frac{7}{8} - \frac{3}{8}\left(\frac{1}{2}\right)$ f. $n = 15\frac{2}{3} - 12\frac{1}{6}\left(3\frac{1}{2}\right)$ i. $n = 8\frac{1}{4} - 5\frac{2}{5}$ $\left(2\frac{17}{20}\right)$

c. $n = \frac{7}{1} - \frac{1}{3}\left(6\frac{2}{3}\right)$ g. $n = 13\frac{5}{8} - 11\left(2\frac{5}{8}\right)$ j. $n = \frac{9}{10} - \frac{1}{2}$ $\left(\frac{2}{5}\right)$

d. $n = \frac{1}{3} - \frac{1}{4}\left(\frac{1}{12}\right)$

8. **Find the numbers** n, p, w, and so on. Express the numbers in simplest form.

a. $\frac{3}{2} + 1 - \frac{1}{4} = n \left(n = 2\frac{1}{4}\right)$

b. $\frac{1}{3} + \frac{4}{6} - 1 = p \left(p = 0\right)$

c. $\frac{1}{3} + \frac{1}{4} - \frac{1}{6} = w \left(w = \frac{5}{12}\right)$

d. $2 - \frac{5}{4} + \frac{1}{2} = s \left(s = 1\frac{1}{4}\right)$

e. $3 - 1\frac{1}{2} + \frac{3}{2} = m \left(m = 3\right)$

f. $9\frac{5}{8} + \frac{6}{16} - \frac{1}{2} = x \left(x = 9\frac{1}{2}\right)$

g. $26\frac{1}{3} + 23\frac{4}{6} - 12\frac{1}{2} = y \left(y = 37\frac{1}{2}\right)$

h. $45\frac{7}{8} - 19\frac{3}{4} - 26\frac{1}{8} = z \left(z = 0\right)$

USING PARENTHESES

Objective: To learn how parentheses are useful in helping us write what we mean.

Vocabulary: Parentheses.

Suggested Teaching Procedure:

Although your pupils may be familiar with the use of parentheses, these pages are for use if needed at this time. Or, you may wish to use them to re-emphasize the use of parentheses.

This lesson can be motivated by writing several mathematical sentences on the chalkboard, such as,

$$2 \times 4 + 3 = n$$

$$8 - 5 + 1 = n, \text{ etc.}$$

and ask pupils to find n for each sentence. (Hope you have different answers.)

Then write again,

$$(2 \times 4) + 3 = n$$

$$8 - (5 + 1) = n, \text{ etc.}$$

and ask that they find n for each sentence. (Now, hope there is agreement.)

Then ask what made the difference?

Emphasize the use of parentheses expressing the name of a number using addition, subtraction, multiplication, division, etc. That is $(3 + 2)$ for 5, (8×2) for 16, etc.

Summarize this discussion by using the exploration in pupil text before pupils do the Exercise Set 5.

USING PARENTHESES

Very early in your study of mathematics you learned that a number can have many names. 7 + 1, 2 × 4, 14 - 6, and 16 ÷ 2 are all other names for 8.

Did you realize, however, that all of the different names for a number must be names for just that one number?

For what number is $6 + 3 \times 4$ another name? Is the number 36 or 18 ?

To remove any doubt about what one number $6 + 3 \times 4$ names, very helpful symbols called parentheses are used.

Notice that $(6 + 3) \times 4$ and $6 + (3 \times 4)$ represent two different numbers.

$$(6 + 3) \times 4 = 9 \times 4 = 36$$

$$6 + (3 \times 4) = 6 + 12 = 18$$

The use of parentheses is very helpful in writing correctly the mathematical sentences for story problems.

Exercise Set 5

1. Which of the following pairs of numerals name the same number? (a)
   - a. $(3 + 2) + 5$ and $3 + (2 + 5)$
   - b. $(16 \div 8) \div 2$ and $16 \div (8 \div 2)$
   - c. $(15 - 3) - 2$ and $15 - (3 - 2)$
   - d. $2 \times (4 + 5)$ and $(2 \times 4) + 5$
   - e. $\frac{3}{4} - (\frac{1}{2} + \frac{1}{4})$ and $(\frac{3}{4} - \frac{1}{2}) + \frac{1}{4}$
2. Place parentheses in the following so that
   - a. $2 \times (3 + 1) = 8$
   - b. $2 + (4 \times 3) = 14$
   - c. $(6 \times 3) - 1 = 17$
   - d. $(12 - 1) \times 2 = 22$

3. Write in numerals, using parentheses.

   a. Subtract the sum of $2\frac{2}{7}$, $\frac{3}{4}$, and $3\frac{1}{2}$ from 10. $(3\frac{13}{28})$

   b. Divide the product of 32 and 67 by 16. (134)

   c. Add 5×8 to the product of 4 and 7. (68)

   d. Divide 2750 by 5 and multiply the result by 3. (1650)

Exercise Set 6

Read each problem carefully. Then write the relationships in the problem as a mathematical sentence. Solve, and write the answer in a complete sentence.

1. Sue and Tom are twins. Sue is $46\frac{3}{4}$ inches tall. Tom is $48\frac{1}{2}$ inches tall. How much taller is Tom than Sue?
   ($46\frac{3}{4} + n = 48\frac{1}{2}$ Tom is $1\frac{3}{4}$ inches taller.)

2. Mary walks $\frac{7}{8}$ of a mile to school. Jane walks $\frac{3}{4}$ of a mile to school. How much farther does Mary walk than Jane? ($\frac{7}{8} - \frac{3}{4} = n$ Mary walks $\frac{1}{8}$ mile farther.)

3. In Mrs. Hardgrove's class $\frac{1}{5}$ of the class goes home for lunch and $\frac{1}{3}$ of the class eats in the cafeteria. The other boys and girls eat bag lunches in the room. What part of the class eats in the room?
   ($\frac{1}{5} + \frac{1}{3} + p = 1$ $\frac{7}{15}$ of the class eats in the room.)

4. Hale's record shop had a "$\frac{1}{4}$ off the original price" sale. What part of the original price did each record cost? ($\frac{4}{4} - \frac{1}{4} = s$ Records cost $\frac{3}{4}$ the original price.)

5. Peggy made a two piece playsuit for herself. The pattern required $\frac{3}{4}$ yards material for the blouse and $1\frac{1}{2}$ yards for the skirt. How much material was required for the playsuit? ($\frac{3}{4} + 1\frac{1}{2} = w$ The playsuit required $2\frac{1}{4}$ yd.)

6. For lunch, Ted ate $\frac{3}{4}$ of a peanut butter sandwich and $\frac{1}{2}$ of a jam sandwich. How many sandwiches did Ted eat for lunch? ($\frac{3}{4}+\frac{1}{2}=n$ Ted ate $1\frac{1}{4}$ sandwiches for lunch.)

7. $\frac{1}{4}$ of the student body of Oaks Junior High School attended Ward Elementary School. $\frac{5}{8}$ of the student body attended Morgan Elementary School. What part of the student body attended elementary schools other than those mentioned?
($1-(\frac{1}{4}+\frac{5}{8})=s$ $\frac{1}{8}$ of the student body attended other schools.)

8. Mrs. Green used $\frac{1}{4}$ of a dozen eggs in a cake and $\frac{1}{6}$ of a dozen eggs in a salad dressing. What part of a dozen eggs did she have left?
($1-(\frac{1}{4}+\frac{1}{6})=y$ Mrs. Green had $\frac{7}{12}$ dozen eggs left.)

9. Bob had a piece of balsa wood one foot long. He cut off two pieces $\frac{1}{2}$ and $\frac{1}{3}$ foot long for the model he was making. How many inches long was the piece he had left?
($1-(\frac{1}{2}+\frac{1}{3})=w$ Bob had a piece 2 inches long left.)

10. Janet filled $\frac{1}{4}$ of her stamp book with American stamps and $\frac{2}{3}$ of the book with stamps from other countries. What part of the book was not filled?
($1-(\frac{1}{4}+\frac{2}{3})=r$ $\frac{1}{12}$ of the book was not filled)

11. If you attend school 9 months of the year, what part of the year are you not in school?
($1-\frac{9}{12}=n$ You are not in school $\frac{1}{4}$ of the yeer.)

12. Alice weighed $69\frac{1}{4}$ pounds at the end of June and $71\frac{2}{3}$ pounds at the end of July. She gained $\frac{1}{2}$ pound in August. How much did Alice gain in July and August together?
($71\frac{2}{3}-69\frac{1}{4}+\frac{1}{2}=p$ Alice gained $2\frac{11}{12}$ pounds.)

DECIMAL NAMES FOR RATIONAL NUMBERS

Objectives: To recall the decimal system of notation, including its application in naming rational numbers.

To name measures of regions which are parts of unit square regions, using decimals.

To find fractions and decimals which are names for the same rational number.

Materials: Pocket chart and cards for place value, Page 88 cards M-W, Pages 83-84; number line J, Page 86.

Vocabulary: Decimal, place value, digit, the fraction form.

Suggested Teaching Procedure:

If necessary, review the decimal place-value system of notation.

Use Cards M, N, and O, and ask the pupils to indicate by decimals the measures of the shaded regions. Use Card P, and suggest that the shaded squares can be rearranged to fill one column (tenth), with five more squares (hundredths). Relate this new arrangement to the decimal 0.15 = 0.1 + 0.05. Continue with Cards R, S, T, U, and V.

Number line J can be used to locate points named by decimal numerals, 0.5, 1.2, and so on. You may wish to draw on the board a number line with a scale of tenths labeled with decimals. Then consider placing a point to be labeled 0.01, and label the tenths scale in hundredths, 0.10, 0.20, etc. Then locate points for 0.52, 1.38, and the like.

The number line can be of assistance to the pupils in ordering rational numbers named by decimals. Locating points for pairs of numbers, 0.5 and 0.14, for example, helps them to decide which of the two is the greater number.

Note the use of a term which may not be familiar--the fraction form of a decimal.

A rational number has many decimal names and many fraction names; for example, the number $0.25 = \frac{25}{100} = \frac{5}{20} = \frac{1}{4} = \frac{50}{200} = 0.250 = 0.2500$, etc.

The purpose of this lesson is to develop an easy way to find one special fraction name for a number named by a decimal. This is done by observing that the digits in the decimal indicate the numerator of a fraction whose denominator is indicated by the place value of the last digit on the right. Consider the decimal 1.52. The digits are 152, so the numerator of the fraction is 152. The last digit, 2, is in hundredths place, so the denominator is 100. Thus $1.52 = \frac{152}{100}$, $15.2 = \frac{152}{10}$, $152. = \frac{152}{1}$, $0.152 = \frac{152}{1000}$.

In each case we have found a fraction name for the number named by the decimal, but we cannot say we have found the fraction name, since each number has many others. For example, $1.52 = \frac{152}{100} = \frac{76}{50} = \frac{456}{300}$.

We call the fraction whose denominator is indicated by the place value of the last digit in the decimal the fraction form of the decimal. So the fraction form of the decimal 1.52 is $\frac{152}{100}$.

DECIMAL NAMES FOR RATIONAL NUMBERS

As you know, any rational number has many fraction names. Certain numbers can also have decimal names. This is true provided the denominator of the fraction name is 10, or 100, or 1,000.

Do you recall the meaning of numerals like 23.64 ? Think of the place value system of numeration.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
		2	3.	6	4	

23.64 means 2 tens + 3 ones + 6 tenths + 4 hundredths or

$23.64 = (2 \times 10) + (3 \times 1) + (\frac{6}{10}) + (\frac{4}{100})$

Look at the last two terms.

$$\begin{aligned} \frac{6}{10} + \frac{4}{100} &= \frac{60}{100} + \frac{4}{100} \\ &= \frac{64}{100} \end{aligned}$$

So you read 23.64 as "twenty-three and sixty-four hundredths". You can write it $23\frac{64}{100}$.

$$\begin{aligned} \text{Since } 23\frac{64}{100} &= 23 + \frac{64}{100} \\ &= \frac{2300}{100} + \frac{64}{100} \\ &= \frac{2364}{100}, \\ 23.64 &= \frac{2364}{100} \end{aligned}$$

Do you see that you can find a fraction name for 23.64 in the following way?

The digits 2364 indicate the numerator.

The place value of the last digit in the decimal 23.64 indicates the denominator.

We shall call the fraction found in this way the fraction form of the decimal.

Now consider the decimal 4.206. What is the numerator of its fraction form? (4206) What is the place value of the digit 6 ? (thousandths) What is the denominator of its fraction form? (1000) How is the fraction form for 4.206 written? $\left(\frac{4206}{1000}\right)$

Exercise Set 7

1. Which of the following decimals have fraction forms with the same numerator? (a, d, g and j ; b, f, and i ; c, h, and e)

    a. 0.13
    b. 2.5
    c. 7.85
    d. 0.013
    e. 78.5
    f. 0.25
    g. 1.3
    h. 0.785
    i. 0.025
    j. 13

2. Which of the decimals in Exercise 1 all have fraction forms with the same denominator? (a, c, and f ; b, e, and g ; d, h, and i)

3. Are there any two of the numerals listed in Exercise 1 that are names for the same number? (No)

4. Write fraction names:

   a. 0.32 $\left(\frac{32}{100}\right)$

   b. 18.04 $\left(\frac{1804}{100}\right)$

   c. 0.075 $\left(\frac{75}{1000}\right)$

   d. 462.5 $\left(\frac{4625}{10}\right)$

   e. 9.1 $\left(\frac{91}{10}\right)$

5. Write decimal names:

   a. $\frac{3}{10}$ (0.3)

   b. $\frac{48}{100}$ (0.48)

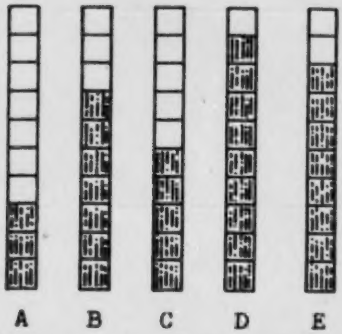
   c. $\frac{132}{100}$ (1.32)

   d. $\frac{38}{1000}$ (0.038)

   e. $\frac{78}{10}$ (7.8)

Exercise Set 8

Each of regions A, B, C, D, and E is a unit region.



1. a. Write the fraction that represents the measure of each shaded region. $\left(A\ \frac{3}{10}\right)\left(B\ \frac{7}{10}\right)\left(C\ \frac{5}{10}\right)\left(D\ \frac{9}{10}\right)\left(E\ \frac{8}{10}\right)$

   b. Write the decimal that represents the measure of each shaded region. (A 0.3)(B 0.7)(C 0.5)(D 0.9)(E 0.8)

2. a. E suggests the addition sentence $\frac{8}{10} + \frac{2}{10} = \frac{10}{10}$.

What addition sentences are suggested by A, B, C, and D ? (A $\frac{3}{10} + \frac{7}{10} = \frac{10}{10} = 1$ B $\frac{7}{10} + \frac{3}{10} = \frac{10}{10} = 1$ C $\frac{5}{10} + \frac{5}{10} = \frac{10}{10} = 1$ D $\frac{9}{10} + \frac{1}{10} = \frac{10}{10} = 1$)

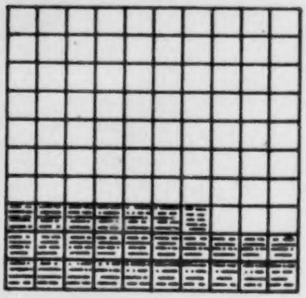
b. E suggests the subtraction sentence $\frac{10}{10} - \frac{8}{10} = \frac{2}{10}$.

What subtraction sentences are suggested by A, B, C, and D ? (A $\frac{10}{10} - \frac{3}{10} = \frac{7}{10}$ B $\frac{10}{10} - \frac{7}{10} = \frac{3}{10}$ C $\frac{10}{10} - \frac{5}{10} = \frac{5}{10}$ D $\frac{10}{10} - \frac{9}{10} = \frac{1}{10}$)

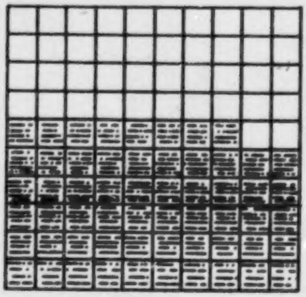
c. Use decimals to write the subtraction sentence suggested by E. Suggested by A, B, C, and D.
(E $1.0 - 0.8 = 0.2$ A $1.0 - 0.3 = 0.7$ B $1.0 - 0.7 = 0.3$ C $1.0 - 0.5 = 0.5$ D $1.0 - 0.9 = 0.1$)

3.

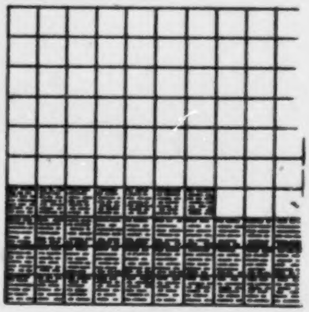
F



G



H



F, G, and H are unit regions.

a. Write the fraction that represents the measure of each shaded region. (F $\frac{27}{100}$) (G $\frac{58}{100}$) (H $\frac{37}{100}$)

b. Write the decimal that represents the measure of each shaded region. (F 0.27) (G 0.58) (H 0.37)

4. What addition sentences are suggested by F, G, and H ?

Write them a) using fractions and b) using decimals.
($\frac{27}{100} + \frac{73}{100} = \frac{100}{100}$; $\frac{58}{100} + \frac{42}{100} = \frac{100}{100}$; $\frac{37}{100} + \frac{63}{100} = \frac{100}{100}$)($0.27 + 0.73 = 1.00$; $0.58 + 0.42 = 1.00$; $0.37 + 0.63 = 1.00$)

5. Write the subtraction sentences suggested by F, G, and H.

Write them a) using fractions and b) using decimals.
($\frac{100}{100} - \frac{27}{100} = \frac{73}{100}$; $\frac{100}{100} - \frac{58}{100} = \frac{42}{100}$; $\frac{100}{100} - \frac{27}{100} = \frac{63}{100}$)($1.00 - 0.27 = 0.73$; $1.00 - 0.58 = 0.42$; $100 - 0.37 = 0.63$)

6. Describe a region you would shade to show 0.001 of the unit square F. (Separate a small square into 10 congruent regions and shade one of them)

7. Describe a region you would shade to show 0.005 of the unit square G. (Regions similar to those for #6. Shade 5 smaller regions.)

8. For each decimal, write the numerator of a fraction form.

a. 0.4 (4) d. 0.1 (1) g. 7.25 (725)

b. 0.25 (25) e. 0.37 (37) h. 13.28 (1328)

c. 0.01 (1) f. 1.8 (18) i. 4.251 (4251)

9. What is the denominator of the fraction form for each decimal in Exercise 8? (a 10 e 100 i 1000
b 100 f 10
c 100 g 100
d 10 h 100)

10. Write as decimals:

a. $\frac{45}{100}$ (0.45) d. $4\frac{38}{100}$ (4.38) g. $\frac{75}{100}$ (0.75)

b. $2\frac{7}{10}$ (2.7) e. $\frac{5}{100}$ (0.05) h. $\frac{1457}{1000}$ (1.457)

c. $\frac{9}{10}$ (0.9) f. $\frac{235}{100}$ (2.35) i. $1\frac{50}{1000}$ (1.050)

11. Write as fractions:

a. 0.65 $\left(\frac{65}{100}\right)$ d. 0.3 $\left(\frac{3}{10}\right)$ g. 0.10 $\left(\frac{10}{100}\right)$

b. 0.8 $\left(\frac{8}{10}\right)$ e. 0.07 $\left(\frac{7}{100}\right)$ h. 7.87 $\left(\frac{787}{100}\right)$

c. 1.70 $\left(\frac{170}{100}\right)$ f. 1.1 $\left(\frac{11}{10}\right)$ i. 0.123 $\left(\frac{123}{1000}\right)$

12. Find n in each sentence:

a. $\frac{3}{4} + 2\frac{5}{6} + 3\frac{2}{3} = n$ $\left(7\frac{1}{4}\right)$ c. $2\frac{1}{2} + \frac{4}{5} + 1 = n$ $\left(4\frac{3}{10}\right)$

b. $3\frac{2}{3} + 2\frac{3}{4} + 1\frac{1}{2} = n$ $\left(7\frac{11}{12}\right)$ d. $6\frac{3}{10} + 11\frac{1}{4} + \frac{4}{5} = n$ $\left(18\frac{7}{20}\right)$

13. Arrange in order from least to greatest:

0.52, 0.056, 1.04, 0.09, 3.68, 0.1, 4.00
(0.056, 0.09, 0.1, 0.52, 1.04, 3.68, 4.00)

14. **Find n:**

a. $n = 14\frac{1}{2} - 7\frac{5}{8}$ $\left(6\frac{7}{8}\right)$

b. $2 - 1\frac{1}{2} = n$ $\left(\frac{1}{2}\right)$

c. $n = 4\frac{1}{4} - 2\frac{5}{6}$ $\left(1\frac{5}{12}\right)$

d. $n = \frac{1}{2} - \frac{1}{9}$ $\left(\frac{7}{18}\right)$

e. $n = 7 - \frac{4}{5}$ $\left(6\frac{1}{5}\right)$

f. $n = 24\frac{2}{3} - 19\frac{3}{4}$ $\left(4\frac{11}{12}\right)$

g. $79\frac{7}{8} - 25\frac{3}{8} = n$ $\left(54\frac{1}{2}\right)$

h. $52\frac{1}{3} - 46\frac{5}{6} = n$ $\left(5\frac{1}{2}\right)$

i. $104\frac{2}{5} - 93\frac{2}{3} = n$ $\left(10\frac{11}{15}\right)$

j. $\frac{1}{2} - \frac{3}{10} = n$ $\left(\frac{1}{5}\right)$

15. **Find t:**

a. $t = 0.9 - 0.71$ (0.19)

b. $t = 0.72 - 0.395$ (0.325)

c. $t = 0.8 - 0.47$ (0.33)

d. $t = 0.35 - 0.2$ (0.15)

e. $t = 27.53 - 8.9$ (18.63)

f. $16.32 - 3.79 = t$ (12.53)

g. $1.2 - 0.09 = t$ (1.11)

h. $5.65 - 0.3 = t$ (5.35)

i. $9.7 - 3.67 = t$ (6.03)

j. $15 - 7.48 = t$ (7.52)

16. **Add:**

a. 0.5 and 0.39 (0.89)

b. 0.73 and 0.6 (1.33)

c. 14.01 and 1.9 (15.91)

d. 1 and 0.1 (1.1)

e. 2.16 and 7.8 (9.96)

f. 47.1 and 9.072 (56.172)

g. 0.07 and 4.3 (4.37)

h. 20.1 and 0.201 (20.301)

Exercise Set 9

Read the following carefully. Show the relationships in each problem using a number-line diagram. Then answer the question asked in the problem.

1. Jack ran the 50 yard dash in 8.7 seconds. Brian ran the distance in 11.0 seconds. Who won? By how many seconds? (11.0 − 8.7 = n Jack won by 2.3 seconds.)

2. During the first six months of a year, 14.8 inches of rain was recorded. During the next six months, 9.79 inches fell. How much rain fell during the year? (14.8 + 9.79 = t 24.59 inches fell that year.)

3. The Empire State Building in New York City is 1250 feet high. The Statue of Liberty in New York harbor is 305.5 feet high. How much higher is the Empire State Building than the Statue of Liberty? (1250 − 305.5 = v The Empire State Building is 944.5 feet higher.)

4. The average annual rainfall of Louisiana is 57.34 inches. The average annual rainfall for Nevada is 8.6 inches. What is the difference between the annual rainfall averages of these two states? (57.34 − 8.6 = m The difference is 48.74 inches)

5. The normal body temperature is 98.6°. Bill's temperature was 0.8° above normal. What was his temperature? (98.6 + 0.8 = m Bill's temperature was 99.4°)

6. Jeff's garden is $30\frac{3}{4}$ feet long and $17\frac{1}{2}$ feet wide. How many feet of wire will it take to put a fence around it? ($30\frac{3}{4} + 30\frac{3}{4} + 17\frac{1}{2} + 17\frac{1}{2} = y$ The fence will take $96\frac{1}{2}$ ft. of wire)

7. Below are the lengths of four Italian ships:

Leonardo da Vinci	761.2	ft.
Augustus	680.4	ft.
Cristoforo Columbo	700.0	ft.
Guillio Cesare	680.6	ft.

(Leonardo da Vinci
Cristoforo Columbo
Guillio Cesare
Augustus)

List the ships in order, from longest to shortest. Then make up three problems about the lengths of the ships.
Problems will very

8. The highest average annual temperature for the world was 88°F. recorded in Africa. The highest average annual temperature for the United States was 77.6°F. recorded in Florida. What is the difference between these two temperatures? (88 - 77.6 = z The difference is 10.4°F.)

9. Pat rode his bicycle $19\frac{1}{3}$ miles one day and $15\frac{2}{5}$ miles the next day. How much farther did he ride his bicycle the first day than the second? ($19\frac{1}{3} - 15\frac{2}{5} = n$ Pat rode $3\frac{14}{15}$ miles farther the first day.)

10. 3.2 inches of rain fell on Monday, 3.0 inches on Tuesday, and 2.4 inches on Wednesday. How many inches of rain fell on the three days? (3.2 + 3.0 + 2.4 = x 8.6 inches of rain fell on these days.)

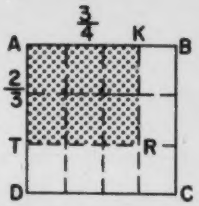
11. The British ship, Queen Elizabeth, is 1031 feet long. The Andes, another British ship, is 669.3 feet long. How much longer is the Queen Elizabeth? (1031.0 - 669.3 = r The Queen Elizabeth is 361.7 ft. longer.)

12. The equatorial diameter of the world is 7,926.68 miles. The polar diameter is 7,899.99 miles. How much greater is the equatorial diameter than the polar diameter?
(7,926 68 - 7,899.99 = s The equatorial diameter is 26.69 miles longer)

Overview of the Next Few Lessons.

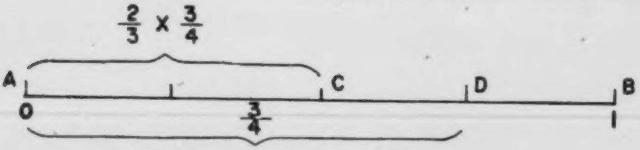
Before suggesting teaching procedures for the lesson of immediate concern, let us first outline briefly our plan for developing an understanding of the multiplication of rational numbers. Three different kinds of physical models are to be used in each of three related developmental lessons. These are models of (1) rectangular regions, (2) line segments on the number line, and (3) collections of groups of objects. The developmental lessons will follow that sequence. For example, we will see how we can associate the product $\frac{2}{3} \times \frac{3}{4}$,

(1) with rectangular regions:



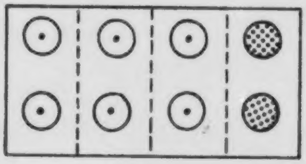
$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ is associated with the shaded region.

(2) with line segments on the number line:



$\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$ is associated with the measure of line segment $\overline{AC}$.

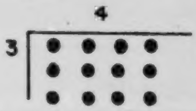
(3) with a collection of objects:



$\frac{3}{4} \times 8 = 6$ is associated with the number of unshaded rings. Also, considering the set as a unit, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{8}$ is associated with the number of rings with dots.

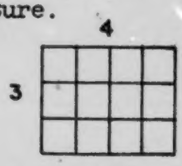
Associating a Product with a Rectangular Region.

The first model used for developing the multiplication of rational numbers is the rectangular region. This choice is made in part because of its similarity to the array which was used to develop the concept of multiplication of whole numbers. You will recall that the product 3×4 was represented by an array of 3 rows and 4 columns.



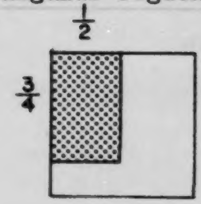
$3 \times 4 = 12.$

In a rectangular region, the measures of whose sides are whole numbers, we have for a 3 by 4 region this picture.

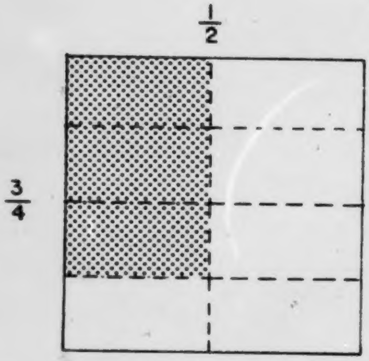


Separating the rectangular region into unit square regions, we again have $3 \times 4 = 12$. That is, the measure of the region is the product of the measures of the sides.

We now assume that the same relation between measures of sides and measure of a rectangular region should be true when the measures of the sides are rational numbers. Therefore, we begin our study of multiplication of rational numbers by examining a rectangular region contained in a unit square region.



The shaded region is a $\frac{3}{4}$ by $\frac{1}{2}$ region. What part of the unit square region is the shaded region? We separate the unit square region by drawing lines through the $\frac{1}{4}$ division marks on one side and the $\frac{1}{2}$ division marks on the other, thus separating the unit square region into 4×2, or 8 congruent rectangular regions. Each of these is $\frac{1}{8}$ of the unit square region. The shaded region covers 3×1, or 3 of these, so the shaded region is $\frac{3}{8}$ of the unit square region, and the measure of the shaded region is $\frac{3}{8}$. After investigating a number of similar examples, we associate the measure of the region with the product of the measures of the sides.



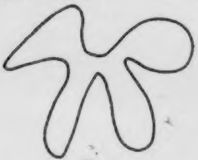
RECTANGULAR REGIONS (REVIEW)

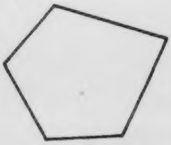
Objective: To review the properties of a rectangular region.

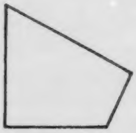
Vocabulary: Simple closed curve, rectangle, square, interior, rectangular region, right angle.

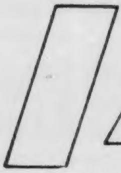
Since understanding of the measure of a region, and specifically, of a rectangular region whose sides have measures which are whole numbers, is basic to the development of multiplication of rational numbers described in the preceding section, the purpose of this lesson is to assure that the pupils have these basic understandings.

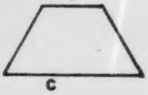
First recall what is meant by (a) a simple closed curve, (b) a simple closed curve which is the union of line segments, (c) a simple closed curve which is the union of four line segments, and finally, (d) a simple closed curve which is the union of four line segments and which has four right angles. Note that in the rectangle, opposite sides are congruent. Some rectangles have all four sides congruent, and are called squares.


a


b


c

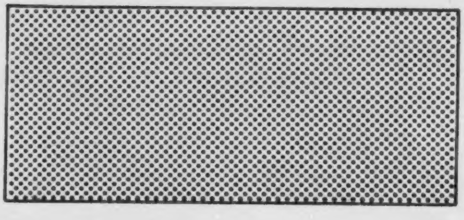

c


c

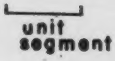
rectangles d

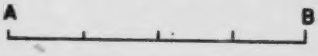
d squares

A rectangular region is the union of the rectangle and its interior.



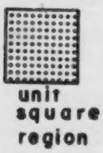
Recall that to measure a segment, we use a unit segment. If we choose the unit segment shown,

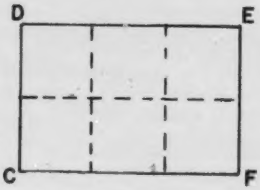




Then the measure of segment $\overline{AB}$ (written m$\overline{AB}$) is 4, since $\overline{AB}$ is the union of 4 segments, each congruent to the unit segment. Note that the measure of $\overline{AB}$ is a number.

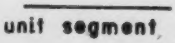
To measure a region, we use as a unit square region. If the unit square region shown below is chosen,

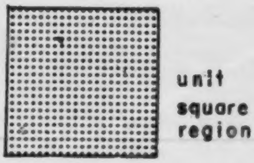




then the measure of the rectangular region CDEF is 6, since the region CDEF can be separated into 6 square regions, each congruent to the unit square region.

In dealing with rectangular regions, we choose a unit square region whose side is a unit segment.





When the sides of a rectangular region are measured using the unit segment, and the region is measured using the related unit square region, we note that the product of the measures of the sides is the measure of the region.

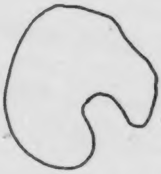
For example, if the sides are 3 inches and 5 inches in length, the measure of the region, in square inches, is 15.

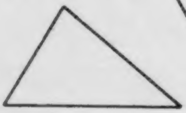
Care should be taken to distinguish the measure of the rectangular region, for which a unit square region is used, and the perimeter of the rectangle, which is the sum of the lengths of the sides, and is therefore found by using a unit segment.

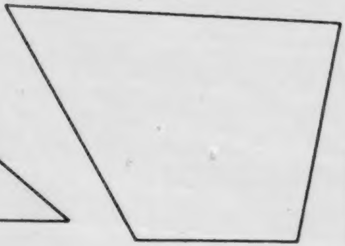
RECTANGULAR REGIONS: REVIEW

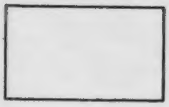
We are going to use rectangular regions to find a way to multiply rational numbers. Do you remember all you learned about rectangular regions?

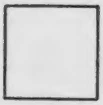
Do you recall what a rectangle is? It is a simple closed curve which is the union of 4 segments and has 4 right angles.


A


B


C


D


E

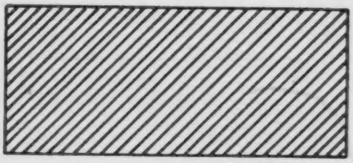
Figures A, B, C, D, and E all represent simple closed curves.

Which figures are the union of four segments? (C, D, and E)

Which figures have four right angles? (D and E)

Figure D represent a rectangle. The union of the rectangle and its interior is a rectangular region.

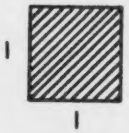
D



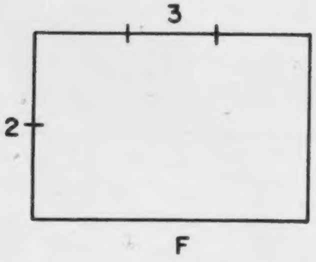
Rectangle D and the shaded part make up a rectangular region.

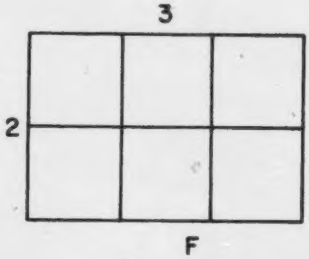
Figure E also represents a rectangle. It is also a square. Why? (4 line segments are congruent.)

Do you recall what kind of region is usually used to measure a rectangular region? It is customary to use a square region with sides 1 unit long.



Suppose Figure F is a rectangle, with sides 3 units and 2 units in length. How do you find the measure of the rectangular region? (2 x 3 = 6)





By drawing lines, we can separate the rectangular region into 6 congruent square regions, each having sides 1 unit in length. These 6 square regions "cover" the rectangular region, so the measure of region F is 6.

You see there are 2 rows of square regions, with 3 in each row. Or there are 3 columns of square regions, with 2 in each column. So there are 2 × 3 or 3 × 2 square regions.

What is an easy way to find the measure of a rectangular region when the measures of its sides are whole numbers? (Multiply the measure of one side by the measure of the other side)

Exercise Set 10

1. a. Draw a rectangle 4 in. by 3 in. (This means 4 in. long and 3 in. wide.) (See bottom of page)

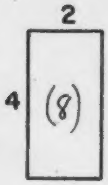
   b. Shade its interior.

   c. Draw lines to separate the rectangular region into unit square regions.

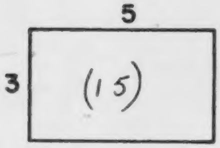
   d. Find the measure of the rectangular region. (12 sq. in)

   e. What is the name of each unit square region? (1 sq. in.)

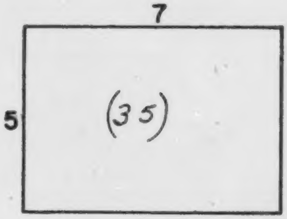
2. Suppose the rectangles A, B, and C below have sides with the measures shown. Find the measures of the rectangular regions.



A

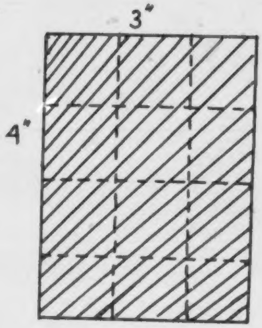


B

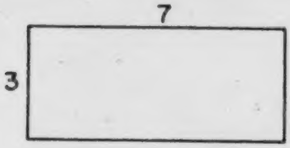


C

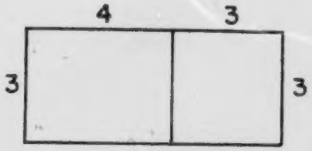
3. Suppose the measure of a rectangular region is 24. What pairs of whole numbers could be the measures of its sides? (6 × 4)(3 × 8)(2 × 12)(1 × 24)



4. Rectangle D has sides whose measures are 7 and 3. What is the measure of the rectangular region? (21)



D



E

5. Figure E is the union of two rectangular regions. Find their measures. Find the sum of their measures. (12 + 9 = 21)

6. Do Exercises 4 and 5 show that

$$3 \times 7 = (3 \times 4) + (3 \times 3) \text{ ?}$$ (Yes)

PRODUCTS AS MEASURES OF RECTANGULAR REGIONS

Objective: To develop an understanding of finding products of rational numbers by using measures of rectangular regions.

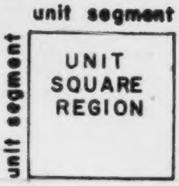
Materials: Cards D, G, H, J, K, and L, page 82

Suggested Teaching Procedure:

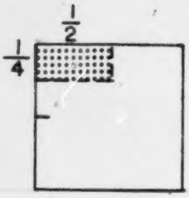
> The pupils have reviewed the relation between the measures of the sides of a rectangle and the measure of the rectangular region, when the measures of the sides were counting numbers. Now propose study of a rectangle the measures of whose sides are rational numbers.

(a) Regions less than a unit square.

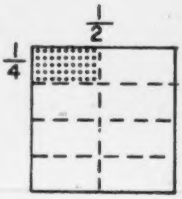
You know how to find the measure of a rectangular region when the sides have measures which are counting numbers. Now suppose you have a rectangle whose sides have measures $\frac{1}{4}$ and $\frac{1}{2}$. We call this a $\frac{1}{4}$ by $\frac{1}{2}$ rectangle. To represent these measures we must start with a unit segment. Let us show this segment as the side of a unit square region. (Sketch A on the board).



A (1st STAGE)



B (2nd STAGE)

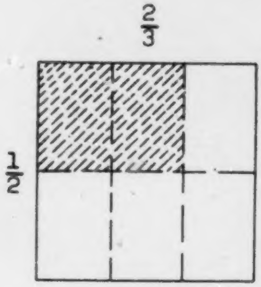


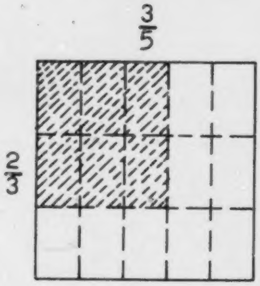
C (3rd STAGE)

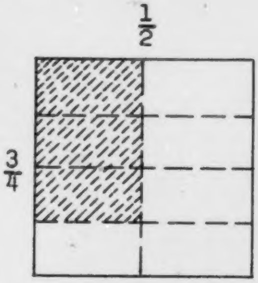
Now let us show the $\frac{1}{4}$ by $\frac{1}{2}$ region in the unit square region. In B, the $\frac{1}{4}$ by $\frac{1}{2}$ rectangular region is shaded. What part of the unit square region is shaded? To find out, draw lines through the $\frac{1}{4}$ and $\frac{1}{2}$ division marks (C). We have now separated the unit square region into eight rectangular regions. Each is congruent to the shaded region. What rational number indicates what part of the unit square region is shaded? What is the measure of the shaded region?

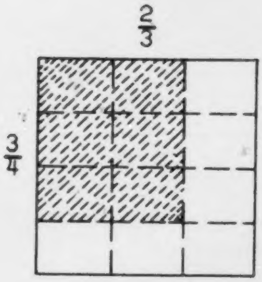
Emphasize that ($\frac{1}{4}$ by $\frac{1}{2}$) and $\frac{1}{8}$ are both associated with the same region.

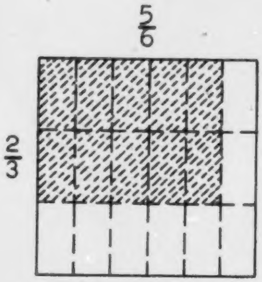
Repeat this exploration using other pairs of rational numbers as measures of the sides of rectangular regions. Here are some suggestions.

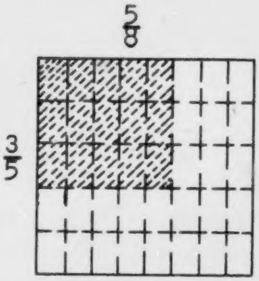










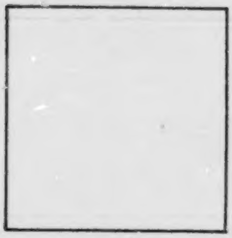


(b) Regions greater than a unit square.

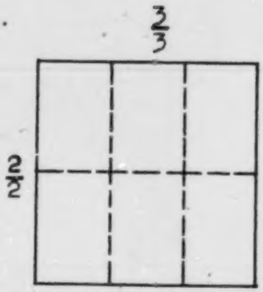
Then suggest rectangular regions whose sides have measures which are rational numbers greater than one. However, name the measures by fractions rather than by mixed forms.

For example, suppose we have a rectangular region whose sides are $\frac{5}{3}$ and $\frac{3}{2}$. Again, start with the unit square.

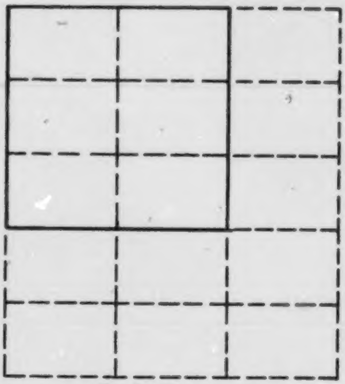
The diagram of this region will be slightly different from others we have made. How shall we separate the unit square region to make the diagram? (Separate one side into three congruent parts and another side into two congruent parts.)



1st Stage



2nd Stage



3rd Stage

The region we have drawn (2nd step) pictures a $\frac{2}{2}$ by $\frac{3}{3}$ region. What is the measure of each of the six small congruent regions? ($\frac{1}{6}$) We wish to represent a $\frac{5}{3}$ by $\frac{3}{2}$ region. What should we do to make one side have measure $\frac{5}{3}$ and the other $\frac{3}{2}$? (Make the region larger.) Tell me how. (Extend the $\frac{3}{3}$ side to make it $\frac{5}{3}$. Extend the $\frac{2}{2}$ side to make it $\frac{3}{2}$.) Now separate the whole region into congruent regions.

The region shown now (3rd stage) is $\frac{5}{3}$ by $\frac{3}{2}$. What is the measure of each of the small congruent regions in the diagram. ($\frac{1}{6}$) How many of them are there? (15)

What is the measure of the $\frac{5}{3}$ by $\frac{3}{2}$ region? ($\frac{15}{6}$)

Have the class summarize their results on the board in a table:

Measures of Sides	Measure of Rectangular Region
$\frac{1}{4}$ by $\frac{1}{2}$	$\frac{1}{8}$
$\frac{1}{2}$ by $\frac{2}{3}$	$\frac{2}{6}$ (Do not simplify)
$\frac{2}{3}$ by $\frac{3}{5}$	$\frac{6}{15}$
etc.	etc.

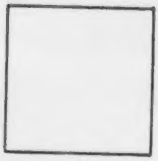
Work through the Exploration with the class, emphasizing that (a) the measures of rectangular regions have been found by using diagrams (Ex. 1-6); (b) the operation used to answer similar questions when the measures were whole numbers was multiplication (Ex. 7); (c) we therefore call the measures of the regions in Ex. 6 the product of the measures of the sides; (d) we can now (Ex. 8) write the mathematical sentences $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$, etc. to show the relation between measures of sides and measure of rectangular region.

PRODUCTS AS MEASURES OF RECTANGULAR REGIONS

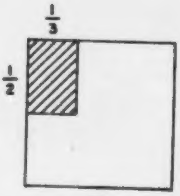
Exploration

You know how to add and subtract rational numbers. Now we shall study multiplication of rational numbers.

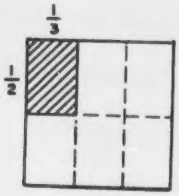
1. Figure A shows a unit square region. Figure B shows the same unit square region. It also shows a shaded rectangular region whose sides are $\frac{1}{2}$ unit and $\frac{1}{3}$ unit in length.



A



B



C

You can separate the unit square region into rectangular regions which are congruent to the $\frac{1}{2}$ by $\frac{1}{3}$ region, as shown in Figure C.

a. How many congruent regions are there? (6)

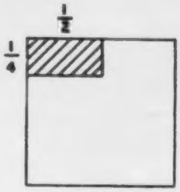
b. What fraction names the measure of the shaded rectangular region? ($\frac{1}{6}$)

2. Figure D shows a $\frac{1}{4}$ by $\frac{1}{2}$ rectangular region shaded. Figure E shows the unit square region separated into congruent rectangular regions.

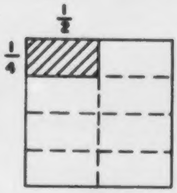
    a. How many rectangular regions congruent to the $\frac{1}{4}$ by $\frac{1}{2}$ region are there in the unit square region? (8)

    b. What is the measure of the $\frac{1}{4}$ by $\frac{1}{2}$ shaded region? $\left(\frac{1}{8}\right)$

A



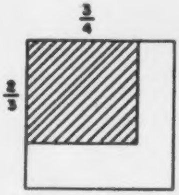
D



E

3. Figure F shows a shaded rectangular region $\frac{2}{3}$ by $\frac{3}{4}$. Figure G shows the shaded rectangular region and also the unit square region separated into congruent rectangular regions.

A



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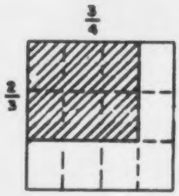


G

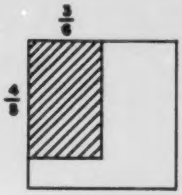
    a. How many small rectangular regions are there in the unit square region? (12)

    b. How many are there in the shaded region? (6)

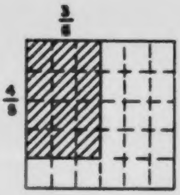
    c. What is the measure of the shaded region? $\left(\frac{6}{12}\right)$

4. Figures H and I show a shaded $\frac{4}{5}$ by $\frac{3}{6}$ rectangular region. What is its measure? $\left(\frac{12}{30}\right)$

A

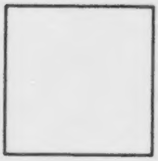


H

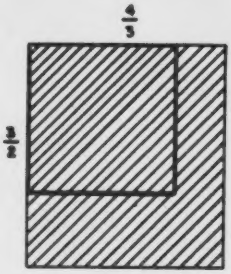


I

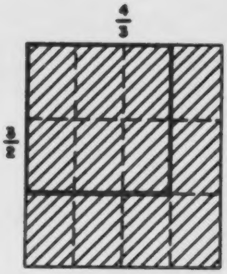
5. Figure J shows a shaded rectangular region larger than the unit square region. The unit square is shown in dark lines. J is a $\frac{3}{2}$ by $\frac{4}{3}$ region. Figure K shows the shaded rectangular region and the unit square region separated into congruent rectangular regions.



A



J



K

a. How many small rectangular regions are there in the unit square region? (6)

b. What is the measure of each small rectangular region? $\left(\frac{1}{6}\right)$

c. How many are there in the shaded region? (12)

d. What is the measure of the shaded region? $\left(\frac{12}{6}\right)$

6. Complete this table about the shaded regions in Exercises 1 - 5.

Measures of Sides	Measure of Region
$\frac{1}{2}$ by $\frac{1}{3}$	$(\frac{1}{6})$
$\frac{1}{4}$ by $\frac{1}{2}$	$(\frac{1}{8})$
$\frac{2}{3}$ by $\frac{3}{4}$	$(\frac{6}{12})$
$\frac{4}{5}$ by $\frac{3}{6}$	$(\frac{12}{30})$
$\frac{3}{2}$ by $\frac{4}{3}$	$(\frac{12}{6})$

7. Now consider some rectangular regions the measures of whose sides are whole numbers. Complete this table.

Measures of Sides	Measure of Region	Operation Used
2 by 3	(6)	(Multiplication)
5 by 8	(40)	(Multiplication)
7 by 6	(42)	(Multiplication)

In Exercise 7 you used multiplication to find the measures of rectangular regions, the measure of whose sides are whole numbers.

We will also call this operation multiplication when the measures of the sides are rational numbers. We will say that

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, because a $\frac{1}{2}$ by $\frac{1}{3}$ region has measure $\frac{1}{6}$.

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, because a $\frac{1}{4}$ by $\frac{1}{2}$ region has measure $\frac{1}{8}$.

P75

8. Write mathematical sentences to show the relation between the measures of the sides and the measure of the region for the other regions in Exercise 6. $\left(\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} \quad \frac{4}{5} \times \frac{3}{6} = \frac{12}{30} \quad \frac{3}{2} \times \frac{4}{3} = \frac{12}{6}\right)$

The measure of a rectangular region whose sides have measures that are rational numbers is the product of those rational numbers.

Exercise Set 11

1. The regions below are unit square regions. The measure of each whole region is 1.

For each shaded region, write

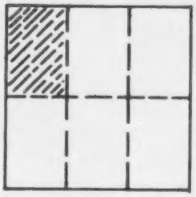
a) the measure of each side.

b) the measure of the region.

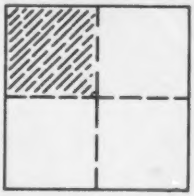
c) a mathematical sentence which shows how the measures of the sides are related to the measure of the region.

Underline the measure of each shaded region.

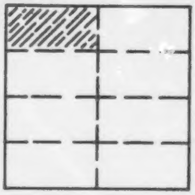
The sentence for A is $\frac{1}{2} \times \frac{1}{3} = \underline{\frac{1}{6}}$.



A



B



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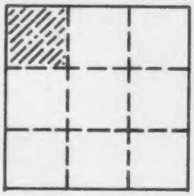

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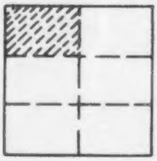
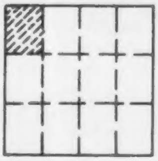


E



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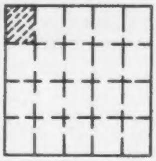

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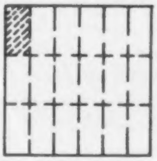
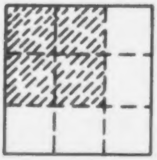
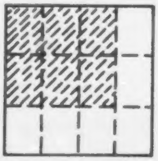


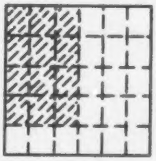
H



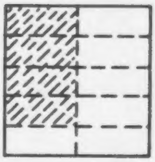
I



J



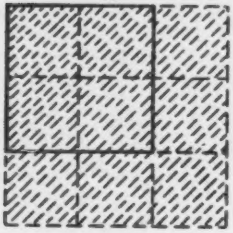
K



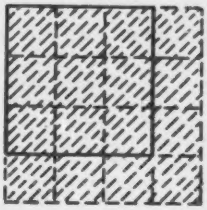
L

2. In each figure below, the unit square is the region bounded by solid lines. For each shaded region, write
   - a) the measure of each side.
   - b) the measure of the region.
   - c) the mathematical sentence which shows the relation of the measures of the sides to the measure of the region. Underline the measure of each shaded region.

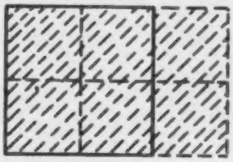
M



N



O



3. Draw a unit square region. Show $\frac{1}{8}$ of this region by drawing lines and shading the region.
4. What mathematical sentence describes the shaded region of Exercise 3 ?
5. Draw a unit square region. Show $\frac{4}{5}$ of this region by drawing lines and shading the region.
6. What mathematical sentence describes the shaded part of Exercise 5 ?
7. Draw a unit square region. Show $\frac{10}{6}$ of this region by drawing lines and shading the region.
8. What mathematical sentence describes the shaded part of Exercise 7 ?

Answers - Exercise Set 11

1. A : a) $\frac{1}{2}$, $\frac{1}{3}$; b) $\frac{1}{6}$; c) $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

B : a) $\frac{1}{2}$, $\frac{1}{2}$; b) $\frac{1}{4}$; c) $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

C : a) $\frac{1}{4}$, $\frac{1}{2}$; b) $\frac{1}{8}$; c) $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

D : a) $\frac{1}{3}$, $\frac{1}{3}$; b) $\frac{1}{9}$; c) $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

E : a) $\frac{1}{3}$, $\frac{1}{2}$; b) $\frac{1}{6}$; c) $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

F : a) $\frac{1}{3}$, $\frac{1}{4}$; b) $\frac{1}{12}$; c) $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

G : a) $\frac{1}{4}$, $\frac{1}{5}$; b) $\frac{1}{20}$; c) $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$

H : a) $\frac{1}{3}$, $\frac{1}{6}$; b) $\frac{1}{18}$; c) $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

I : a) $\frac{2}{3}$, $\frac{2}{3}$; b) $\frac{4}{9}$; c) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

J : a) $\frac{2}{3}$, $\frac{3}{4}$; b) $\frac{6}{12}$; c) $\frac{2}{3} \times \frac{3}{44} = \frac{6}{12}$

K : a) $\frac{4}{5}$, $\frac{3}{6}$; b) $\frac{12}{30}$; c) $\frac{4}{5} \times \frac{3}{6} = \frac{12}{30}$

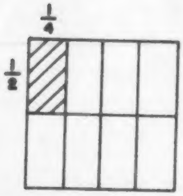
L : a) $\frac{4}{5}$, $\frac{1}{2}$; b) $\frac{4}{10}$; c) $\frac{4}{5} \times \frac{1}{2} = \frac{4}{10}$

2. M : a) $\frac{3}{2}$, $\frac{3}{2}$ b) $\frac{9}{4}$ c) $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$

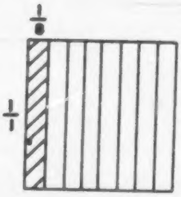
N : a) $\frac{4}{3}$, $\frac{4}{3}$ b) $\frac{16}{9}$ c) $\frac{4}{3} \times \frac{4}{3} = \frac{16}{9}$

O : a) $\frac{2}{2}$, $\frac{3}{2}$ b) $\frac{6}{4}$ c) $\frac{2}{2} \times \frac{3}{2} = \frac{6}{4}$

3.

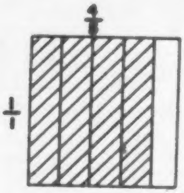


or



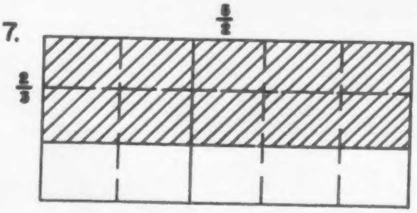
4. $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ or $\frac{1}{1} \times \frac{1}{8} = \frac{1}{8}$

5.



6. $\frac{1}{1} \times \frac{4}{5} = \frac{4}{5}$

7.



8. $\frac{2}{3} \times \frac{5}{2} = \frac{10}{6}$

RATIONAL NUMBERS AND WHOLE NUMBERS

Objective: To consider the product of a pair of rational numbers which are also whole numbers, to find a clue which might lead to a procedure for computing the product of any two rational numbers.

Suggested Teaching Procedure:

Work through the Exploration with the class. Use additional examples if necessary.

If the pupils make the generalization that finding the product of the numerators and the product of the denominators of fraction names for 2 and 3 produces a fraction name for 6, raise this question:

Will this relation be different when pairs of fractions are not names for whole numbers?

RATIONAL NUMBERS AND WHOLE NUMBERS

Exploration

You can use what you know about multiplication of whole numbers to study multiplication of rational numbers. Consider the product $3 \times 2 = 6$.

1. Here are some fraction names for the numbers 2, 3, and 6:

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{20}{10}$$

$$3 = \frac{3}{1} = \frac{6}{2} = \frac{12}{4} = \frac{15}{5}$$

$$6 = \frac{6}{1} = \frac{12}{2} = \frac{18}{3} = \frac{30}{5} = \frac{36}{6}$$

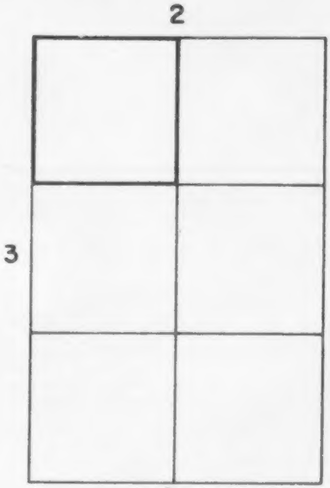
Since $3 \times 2 = 6$, should $\frac{3}{1} \times \frac{2}{1}$ be another name for 6 ? (Yes)

should $\frac{6}{2} \times \frac{4}{2}$ be another name for 6 ? (Yes)

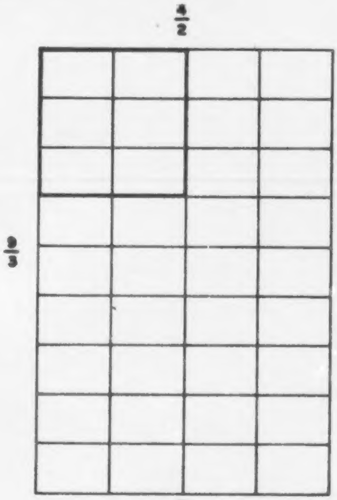
should $\frac{12}{4} \times \frac{6}{3}$ be another name for 6 ? (Yes)

2. a. Figure A is a 3 by 2 rectangular region.

Figure B is a $\frac{9}{3}$ by $\frac{4}{2}$ rectangular region. Should A and B have the same measure? (Yes)



A



B

b. Region A is separated into unit square regions to show that its measure is (3) × (2) or (6).

c. Region B is separated into (36) congruent rectangular regions.

d. The unit square is shown with dark lines. Each small rectangular region is $(\frac{1}{6})$ of the unit square region.

e. From c and d , you know that the measure of region B is $(\frac{36}{6})$.

f. Does Region B show that $\frac{9}{3} \times \frac{4}{2} = \frac{36}{6}$? (Yes)

g. Compare your answers for b and f. Are the measures of regions A and B the same? (Yes) Does $3 \times 2 = \frac{9}{3} \times \frac{4}{2}$? (Yes)

3. Consider $2 = \frac{6}{3}$ and $3 = \frac{6}{2}$.

Since 2 × 3 = 6, should $\frac{6}{3} \times \frac{6}{2}$ be another name for 6 ? (Yes) Try some operations with the numerators and denominators to find a fraction name for 6.

4. Is $\frac{6 + 6}{3 + 2} = 6$ a true statement? (No)

Is $\frac{6 - 6}{3 - 2} = 6$ a true statement? (No)

Is $\frac{6 \div 6}{3 \div 2} = 6$ a true statement? (No)

Is $\frac{6 \times 6}{3 \times 2} = 6$ a true statement? (Yes)

5. Without using a drawing, try to find the product 2 × 3 by using a different pair of fraction names for 2 and 3. Did you find any operation on the numerators and denominators which seemed to give a fraction name for 6 ?
(Answers will vary)

COMPUTING PRODUCTS OF RATIONAL NUMBERS USING FRACTIONS

Objective: To develop a method for computing the product of any two rational numbers by finding the products of counting numbers associated with the fraction names. (numerators and denominators)

Suggested Teaching Procedure:

Review the observations made about measures of sides and of rectangular regions, and the observations made about the product of a pair of rational numbers which are also whole numbers. Ask children if these observations suggest a way of finding the product of any two rational numbers. The series of questions you ask should be for the purpose of motivating children to arrive at the generalization that the product of the numerators will indicate the numerator for a fraction, that the product of the denominators will indicate a denominator for a fraction, and that this fraction indicates the unique rational number that is the product of the two rational numbers.

Continue as needed. Hope that soon children will say, "We do not need to think of diagrams. We can find the numerator of a fraction name for the product by multiplying the numerators of the fractions, and the denominator by multiplying the denominators of the fractions."

Also, observe that using different names for these rational numbers still does not change the product. For example,

$$\frac{3}{4} \times \frac{2}{3} = \frac{(3 \times 2)}{(4 \times 3)} = \frac{6}{12}$$

$$\frac{6}{8} \times \frac{4}{6} = \frac{(6 \times 4)}{(8 \times 6)} = \frac{24}{48}$$

$\frac{6}{12}$ and $\frac{24}{48}$ name the same number.

Observe that we use products of counting numbers which we already know.

Here are some other pairs of rational numbers. Suggest that the pupils find the product without use of diagrams if they can. If some need diagrams, let them use them.

(a) $\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12}$

(b) $\frac{7}{10} \times \frac{3}{10} = n$

(c) $\frac{4}{9} \times \frac{5}{7} = n$

(d) $\frac{6}{11} \times \frac{2}{3} = n$

(e) $\frac{2}{9} \times \frac{3}{5} = n$

(f) $\frac{4}{10} \times \frac{7}{25} = n$

(g) $\frac{3}{5} \times \frac{2}{7} = n$

(h) $\frac{3}{b} \times \frac{2}{7} = n$

(i) $\frac{a}{b} \times \frac{5}{d} = n$

(j) $\frac{a}{b} \times \frac{c}{d} = n$

COMPUTING PRODUCTS OF RATIONAL NUMBERS USING FRACTIONS

Exploration

If your work for Exercise 1 in Exercise Set 11 was correct, you wrote these mathematical sentences:

A. $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ K. $\frac{4}{5} \times \frac{3}{6} = \frac{12}{30}$

H. $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$ M. $\frac{4}{3} \times \frac{3}{2} = \frac{12}{6}$

I. $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ O. $\frac{2}{2} \times \frac{3}{2} = \frac{6}{4}$

1. Look at sentence A again. Does $\frac{1}{6} = \frac{1 \times 1}{2 \times 3}$? (Yes)

   In H: Does $\frac{1}{18} = \frac{1 \times 1}{3 \times 6}$? (Yes) K: Does $\frac{12}{30} = \frac{4 \times 3}{5 \times 6}$? (Yes)

   I: Does $\frac{4}{9} = \frac{2 \times 2}{3 \times 3}$? (Yes) M: Does $\frac{12}{6} = \frac{4 \times 3}{3 \times 2}$? (Yes)

   O: Does $\frac{6}{4} = \frac{2 \times 3}{2 \times 2}$? (Yes)

2. If a and b are any counting numbers, what is the product $\frac{1}{a} \times \frac{1}{b}$? $\left(\frac{1 \times 1}{a \times b} = \frac{1}{a \times b}\right)$

3. If a and c are any whole numbers, and b and d are any counting numbers, what is $\frac{a}{b} \times \frac{c}{d}$? $\left(\frac{a \times c}{b \times d}\right)$

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}}$$

Exercise Set 12

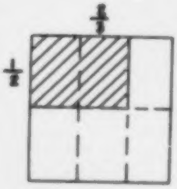
Find a single rational number for the product expressions:

1. $\frac{1}{2} \times \frac{1}{5}$ $\left(\frac{1}{10}\right)$
2. $\frac{1}{5} \times \frac{1}{3}$ $\left(\frac{1}{15}\right)$
3. $\frac{1}{2} \times \frac{1}{4}$ $\left(\frac{1}{8}\right)$
4. $\frac{3}{4} \times \frac{2}{5}$ $\left(\frac{6}{20}\right)$ or $\left(\frac{3}{10}\right)$
5. $\frac{2}{3} \times \frac{3}{4}$ $\left(\frac{6}{12}\right)$ or $\left(\frac{1}{2}\right)$
6. $\frac{4}{5} \times \frac{2}{3}$ $\left(\frac{8}{15}\right)$
7. $\frac{3}{5} \times \frac{4}{5}$ $\left(\frac{12}{25}\right)$
8. $\frac{3}{2} \times \frac{4}{5}$ $\left(\frac{12}{10}\right.$ or $\frac{6}{5}$ or $\left.1\frac{1}{5}\right)$
9. $\frac{6}{5} \times \frac{3}{4}$ $\left(\frac{18}{20}\right.$ or $\left.\frac{9}{10}\right)$
10. $\frac{1}{4} \times \frac{1}{5}$ $\left(\frac{1}{20}\right)$
11. $\frac{1}{5} \times \frac{1}{6}$ $\left(\frac{1}{30}\right)$
12. $\frac{1}{2} \times \frac{1}{6}$ $\left(\frac{1}{12}\right)$
13. $\frac{3}{4} \times \frac{3}{4}$ $\left(\frac{9}{16}\right)$
14. $\frac{2}{3} \times \frac{3}{5}$ $\left(\frac{6}{15}\right.$ or $\left.\frac{2}{5}\right)$
15. $\frac{5}{6} \times \frac{2}{3}$ $\left(\frac{10}{18}\right.$ or $\left.\frac{5}{9}\right)$
16. $\frac{2}{7} \times \frac{1}{3}$ $\left(\frac{2}{21}\right)$
17. $\frac{2}{3} \times \frac{3}{2}$ $\left(\frac{6}{6}\right.$ or $\left.1\right)$
18. $\frac{5}{2} \times \frac{10}{3}$ $\left(\frac{50}{6}\right.$ or $\frac{25}{3}$ or $\left.8\frac{1}{3}\right)$
19. $\frac{2}{5} \times \frac{2}{3}$ $\left(\frac{4}{15}\right)$
20. $\frac{3}{8} \times \frac{3}{4}$ $\left(\frac{9}{32}\right)$
21. $\frac{2}{9} \times \frac{1}{2}$ $\left(\frac{2}{18}\right.$ or $\left.\frac{1}{9}\right)$
22. $\frac{7}{10} \times \frac{2}{3}$ $\left(\frac{14}{30}\right.$ or $\left.\frac{7}{15}\right)$
23. $\frac{2}{5} \times \frac{3}{7}$ $\left(\frac{6}{35}\right)$
24. $\frac{4}{3} \times \frac{5}{3}$ $\left(\frac{20}{9}\right.$ or $\left.2\frac{2}{9}\right)$
25. $\frac{1}{10} \times \frac{1}{10}$ $\left(\frac{1}{100}\right)$
26. $\frac{1}{10} \times \frac{1}{10^2}$ $\left(\frac{1}{1000}\right)$
27. $\frac{1}{10^2} \times \frac{1}{10^2}$ $\left(\frac{1}{10,000}\right)$

Draw rectangular regions to illustrate these products:

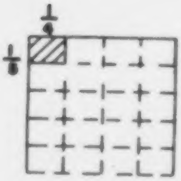
28. $\frac{1}{2} \times \frac{2}{3}$
29. $\frac{1}{5} \times \frac{1}{4}$
30. $\frac{4}{3} \times \frac{6}{5}$
31. $\frac{3}{4} \times \frac{3}{5}$

28.



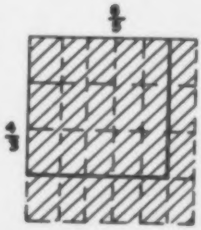
$\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$

29.



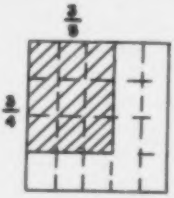
$\frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$

30.



$\frac{4}{3} \times \frac{6}{5} = \frac{24}{15}$

31.



$\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$

Exercise Set 13

1. The table shows some measures of rectangular regions.

Complete the table:

	Measure of One Side	Measure of Other Side	Measure of Rectangular Region
a.	$\frac{2}{3}$	$\frac{2}{3}$	$\left(\frac{4}{9}\right)$
b.	$\frac{5}{4}$	$\frac{3}{4}$	$\left(\frac{15}{16}\right)$
c.	$\frac{2}{5}$	$\frac{3}{4}$	$\left(\frac{6}{20}$ or $\frac{3}{10}\right)$
d.	$\frac{1}{2}$	$\frac{17}{8}$	$\left(\frac{17}{16}$ or $1\frac{1}{16}\right)$
e.	$\frac{2}{3}$	$\frac{1}{2}$	$\left(\frac{2}{6}$ or $\frac{1}{3}\right)$
f.	$\frac{3}{4}$	$\frac{4}{5}$	$\left(\frac{12}{20}$ or $\frac{6}{10}$ or $\frac{3}{5}\right)$
g.	$\frac{2}{5}$	$\frac{3}{7}$	$\left(\frac{6}{35}\right)$
h.	$\frac{a}{b}$	$\frac{c}{d}$	$\left(\frac{a \times c}{b \times d}\right)$

2. Rename each of the following in mixed form or as a whole number:

a. $\frac{4}{3}$ $\left(1\frac{1}{3}\right)$ c. $\frac{16}{9}$ $\left(1\frac{7}{9}\right)$ e. $\frac{12}{6}$ (2)

b. $\frac{3}{2}$ $\left(1\frac{1}{2}\right)$ d. $\frac{2}{2}$ (1) f. $\frac{6}{4}$ $\left(1\frac{1}{2}\right)$

3. Rename each of the following by a fraction in simplest form:

a. $3\frac{1}{2}$ $\left(\frac{7}{2}\right)$ c. $1\frac{1}{2}$ $\left(\frac{3}{2}\right)$ e. $5\frac{2}{2}$ $\left(\frac{6}{1}\right)$

b. $6\frac{2}{3}$ $\left(\frac{20}{3}\right)$ d. $3\frac{3}{4}$ $\left(\frac{15}{4}\right)$ f. $4\frac{3}{5}$ $\left(\frac{23}{5}\right)$

NAMING PRODUCTS WITH DECIMALS

Objectives: To illustrate the use of decimals to indicate measures of the sides of a rectangle and the measure of the rectangular region.

To show that the measure of the region may be found by (a) using a diagram of a unit square region, and (b) using fraction names to compute the product of the measures of the sides.

Suggested Teaching Procedure:

In addition to developing the ideas presented in the Pupil Text, you may also wish to consider the relation of the measure of the 0.7 by 0.8 region found from the diagram in the text to the decimal system of notation. By rearranging the small shaded squares it can be shown that the 56 small squares will cover 5 strips (tenths of the unit square region) and 6 additional small squares (hundredths of the unit square region), or (0.5 + 0.06) of the unit square region.

Formal procedures for computing products of rational numbers using decimal names are developed later in this chapter.

NAMING PRODUCTS WITH DECIMALS

Exploration

1. a. Draw a unit square. (See answer at bottom of page)

   b. Draw dotted lines and show by shading a $\frac{1}{10}$ by $\frac{1}{10}$ region of the unit square.

   c. What is the measure of the shaded region? ($\frac{1}{100}$)

   d. Write a mathematical sentence suggested by your diagram. ($\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$

2. a. Draw a unit square. (See unit square in 1a.)

   b. Draw dotted lines and show by shading a 0.1 by 0.1 region of the unit square.

   c. What is the measure of the shaded region? (0.01)

   d. Write a mathematical sentence suggested by your diagram. ($0.1 \times 0.1 = 0.01$)

3. How are the mathematical sentences you wrote for Exercise 1 and 2 alike? (0.1 and $\frac{1}{10}$ are names for the same number. 0.01 and $\frac{1}{100}$ are names for the same number.)

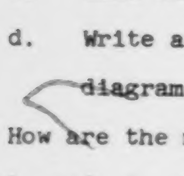
4. Finish these sentences without the use of diagrams.

   a. $\frac{1}{10} \times \frac{1}{10} = \underline{(\frac{1}{100})}$ c. $\frac{1}{10} \times \frac{1}{100} = \underline{(\frac{1}{1000})}$ e. $1 \times \frac{1}{10} = \underline{(\frac{1}{10})}$

   b. $0.1 \times 0.1 = \underline{(0.01)}$ d. $0.1 \times 0.01 = \underline{(0.001)}$ f. $1 \times 0.1 = \underline{(0.1)}$

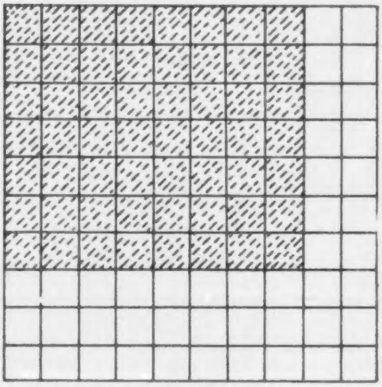
The products you have just found are important for you to remember. You will use them often.

(1.



)

The product 0.7 × 0.8 is represented in the diagram.



The shaded region is a 0.7 by 0.8 region, so its measure is 0.7 × 0.8.

The unit square region is separated into 10 × 10, or 100 congruent square regions. So the measure of each small square region is 0.01. The shaded region covers 7 × 8 or 56 small square regions, so its measure is 56 × 0.01, or 0.56.

$$0.7 \times 0.8 = 0.56$$

The product of the numerators of the fraction forms ($\frac{7}{10}$ and $\frac{8}{10}$) for 0.7 and 0.8 is 7 × 8 or 56. The product of the denominators is 100. A fraction with numerator 56 and denominator 100 names the same number as 0.56.

Exercise Set 14

1. Draw a unit square region. Separate it into 10 × 10, or 100, congruent square regions. Use this unit square region to find the products below. (See Exploration)

a.	0.2 × 0.8 (0.16)	c.	0.9 × 0.7 (0.63)
b.	0.6 × 0.3 (0.18)	d.	0.5 × 0.4 (0.20)

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2. The table shows measurements for rectangular regions. Complete the table.

	Length of One Side	Length of Other Side	Perimeter	Area
a.	$1\frac{1}{2}$ in.	$\frac{3}{4}$ in.	$4\frac{1}{2}$ in.	$1\frac{1}{8}$ sq. in.
b.	1 ft.	$2\frac{1}{3}$ ft.	($6\frac{2}{3}$ ft)	($2\frac{1}{3}$ sq. ft)
c.	$\frac{2}{3}$ ft.	$2\frac{1}{4}$ ft.	($5\frac{5}{6}$ ft)	($1\frac{1}{2}$ sq. ft.)
d.	7 in.	$\frac{5}{8}$ in.	($15\frac{1}{4}$ in.)	($4\frac{3}{8}$ sq. in)
e.	0.7 mi.	0.5 mi.	(2.4 mi)	(0.35 sq mi)
f.	$\frac{3}{4}$ mi.	6 mi.	($13\frac{1}{2}$ mi)	($4\frac{1}{2}$ sq mi)
g.	$7\frac{1}{2}$ yd.	2 yd.	(19 yd)	(15 sq yd)
h.	$\frac{7}{8}$ in.	$\frac{3}{4}$ in.	($3\frac{1}{4}$ in)	($\frac{21}{32}$ sq in)
i.	$\frac{1}{2}$ ft.	$4\frac{1}{3}$ ft.	($9\frac{2}{3}$ ft)	($2\frac{1}{6}$ sq ft)
j.	0.8 in.	0.3 in.	(2.2 in)	(0.24 sq in)
k.	0.12 mi.	0.5 mi.	(1.24 mi)	(0.060 sq mi)
l.	$\frac{1}{2}$ ft.	$8\frac{1}{2}$ ft.	(18 ft)	($4\frac{1}{4}$ sq ft)

PRODUCTS OF RATIONAL NUMBERS USING THE NUMBER LINE

Objective: To use segments on the number line as a model for multiplication of rational numbers.

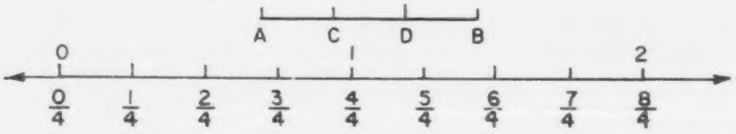
Materials: Number lines, page 86

Vocabulary: Scale.

Suggested Teaching Procedure:

> We have developed the idea of the product of two rational numbers by use of rectangular regions. It would be unfortunate if a child associated the product of two rational numbers with only this one kind of physical model. We turn now to a second model, segments on the number line. This model is very important, because it is a useful model as an aid in problem solving and because we have already associated the rational numbers with points on the number line.

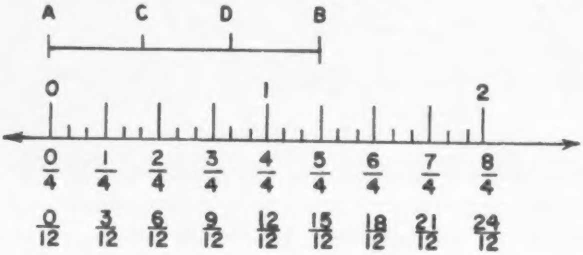
Start by drawing a number line on the chalkboard. Write the whole number scale above and the scale in fourths below.



Identify the unit segment. Ask how we can represent a segment whose measure is $\frac{5}{4}$. (One endpoint should match 0 and the other $\frac{5}{4}$) Draw $\overline{AB}$. $m\overline{AB} = \frac{5}{4}$. (We must separate $\overline{AB}$ into three congruent segments.) Mark C and D to do this. $\overline{AC}$, $\overline{CD}$, and $\overline{DB}$ each have measure $\frac{1}{3}$ of $\frac{5}{4}$. Since $\overline{AC}$ has point A at 0, the number which matches C is $\frac{1}{3}$ of $\frac{5}{4}$.

What number is this? To find out we must separate the unit segment into smaller congruent parts, just as we separated unit square regions. How many should we make? Would separating each segment with measure $\frac{1}{4}$ into three congruent parts do?

Separate each $\frac{1}{4}$ segment on the number line into three congruent segments. Each small segment is what part of the unit segment? $(\frac{1}{12})$ Write the scale of twelfths.



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$\frac{0}{12}$ $\frac{3}{12}$ $\frac{6}{12}$ $\frac{9}{12}$ $\frac{12}{12}$ $\frac{15}{12}$ $\frac{18}{12}$ $\frac{21}{12}$ $\frac{24}{12}$

What number matches C? $(\frac{5}{12})$ Then $m\overline{AC} = \frac{5}{12}$. $\frac{1}{3}$ of $\frac{5}{4} = \frac{5}{12}$.

What segment with endpoint A is $\frac{2}{3}$ of $\overline{AB}$? $(\overline{AD})$

What number does D match? $(\frac{10}{12})$ What is $\frac{2}{3}$ of $\frac{5}{4}$?

Continue with the Exploration. Following Ex. 1-3, the results are summarized. These are compared (Ex. 4) with the products of the same pairs of numbers found by computation. Ex. 5 emphasizes the observation that $\frac{4}{5}$ of $\frac{2}{3}$ as interpreted with segments on the number line yields the same result as $\frac{4}{5} \times \frac{2}{3}$ found by computation and Ex. 6 and 7 reinforce this idea.

In common language, we often speak about $\frac{2}{3}$ of $\frac{3}{4}$ of a thing. For example, we walk $\frac{2}{3}$ of a distance which is known to be $\frac{3}{4}$ of a mile. From the Exploration we can note that such a situation may be properly associated with the operation of multiplication of rational numbers.

PRODUCTS OF RATIONAL NUMBERS USING THE NUMBER LINE

Exploration

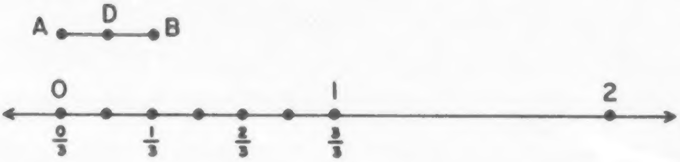
We have used the relation between the measures of the sides and the measure of a rectangular region to give a meaning to the product of any two rational numbers.

If we wish, we can always picture $\frac{1}{2} \times \frac{2}{3}$ as the measure of a $\frac{1}{2}$ by $\frac{2}{3}$ rectangular region. There are, however, other situations which lead to the same rule: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

We shall now study some of these other meanings for the product of rational numbers.

First let us use the number line to think about what we usually mean by "$\frac{1}{2}$ of $\frac{1}{3}$" or "$\frac{4}{5}$ of $\frac{2}{3}$".

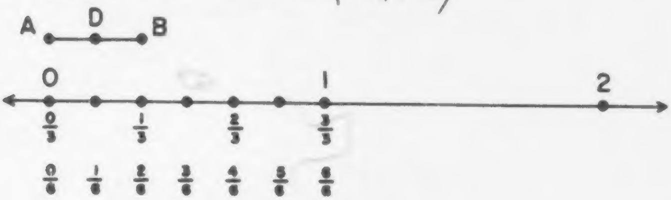
1. Begin by representing $\frac{1}{2}$ of $\frac{1}{3}$. Look at the number line and $\overline{AB}$.



$\overline{AB}$ has measure $\frac{1}{3}$.

To represent $\frac{1}{2}$ of $\frac{1}{3}$, locate D to separate $\overline{AB}$ into congruent segments $\overline{AD}$ and $\overline{DB}$. $\overline{AD}$ is $\frac{1}{2}$ of $\overline{AB}$. So $m\overline{AD}$ should be $\frac{1}{2}$ of $\frac{1}{3}$.

a. $\frac{1}{2}$ of $\frac{1}{3}$ should be a number which matches D. To find this number, you need a scale with a smaller unit. What scale? (sixths)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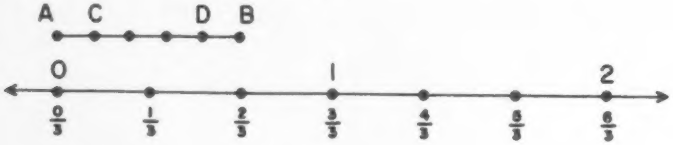
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$

You see that D matches $\frac{1}{6}$, so the diagram shows that $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$.

2. Now consider $\frac{4}{5}$ of $\frac{2}{3}$.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$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$

a. What segment has measure $\frac{2}{3}$? ($\overline{AB}$)

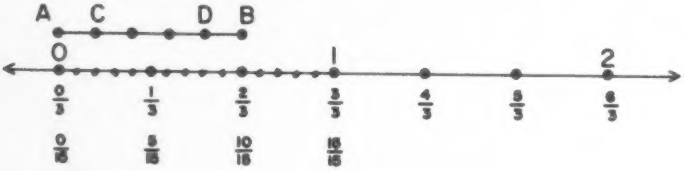
b. How do you find a segment which is $\frac{1}{5}$ of $\overline{AB}$?
(Seperate $\overline{AB}$ into 5 congruent segments.)

c. What segment with endpoint A has measure $\frac{1}{5}$ of $\frac{2}{3}$?
($\overline{AC}$)

d. What segment with endpoint A has measure $\frac{4}{5}$ of $\frac{2}{3}$?
($\overline{AD}$)

e. To find $\frac{4}{5}$ of $\frac{2}{3}$, you must find a number on the number line which matches point (D).

f. You need a scale marked with a smaller unit. What unit? (Fifteenths) Should separating each $\frac{1}{3}$ segment into 5 congruent segments do? (Yes)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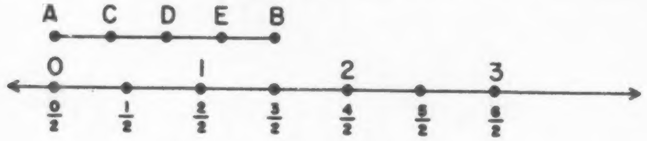
$\frac{6}{3}$
$\frac{0}{15}$
$\frac{5}{15}$
$\frac{10}{15}$
$\frac{15}{15}$

g. What number matches D? $\left(\frac{8}{15}\right)$

h. If $\frac{4}{5}$ of $\frac{2}{3} = n$, what number is n? $\left(\frac{8}{15}\right)$

3. Now find $\frac{7}{4}$ of $\frac{3}{2}$ using the number line.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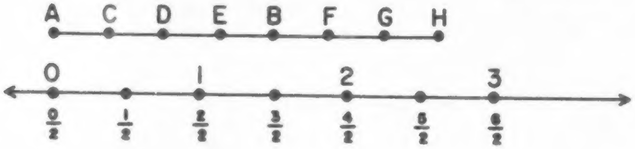
a. $\overline{AB}$ has measure $(\frac{3}{2})$.

b. To represent $\frac{7}{4}$ of $\frac{3}{2}$, first separate $\overline{AB}$ into (4) congruent segments. $\overline{AB}$ is separated into congruent segments by points (C), (D), and (E).

c. What segment with endpoint A has measure $\frac{1}{4}$ of $\frac{3}{2}$? $(\overline{AC})$

d. To represent $\frac{7}{4}$ of $\frac{3}{2}$, you need (7) of these segments. $\overline{AB}$ is not long enough to represent $\frac{7}{4}$ of $\frac{3}{2}$, so draw three more segments: $\overline{BF}$, $\overline{FG}$, and $\overline{GH}$.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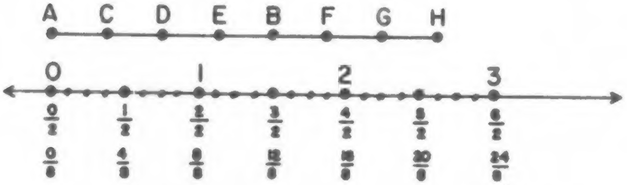
6/2

e. Segment $(\overline{AH})$ has measure $\frac{7}{4}$ of $\frac{3}{2}$.

f. To find $\frac{7}{4}$ of $\frac{3}{2}$, find what number matches point (H).

g. You need a scale in smaller units. A scale of (eighths) will do.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16/8
20/8
24/8

h. What number matches H? $(\frac{21}{8})$

i. $\frac{7}{4}$ of $\frac{3}{2} = n.$ $n = ?$ $(\frac{21}{8})$

Now look at your results for Exercises 1 - 3.

In Exercise 1 you found that $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$.

In Exercise 2 you found that $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$.

In Exercise 3 you found that $\frac{7}{4}$ of $\frac{3}{2} = \frac{21}{8}$.

4. Use the rule $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ to find

a. $\frac{1}{2} \times \frac{1}{3}$ $\left(\frac{1}{6}\right)$

b. $\frac{4}{5} \times \frac{2}{3}$ $\left(\frac{8}{15}\right)$

c. $\frac{7}{4} \times \frac{3}{2}$ $\left(\frac{21}{8}\right)$

How do your products compare with your results for Exercises 1 - 3 ? (They are the same.)

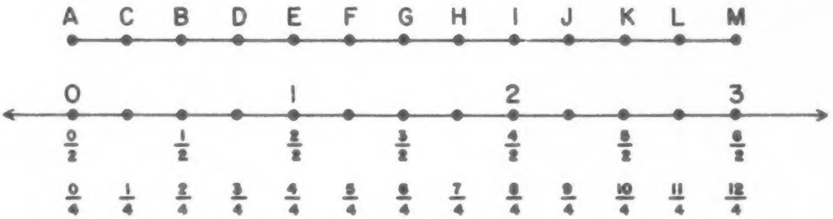
5. Are these reasonable statements? (Yes)

On the number line, $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$

$$\frac{8}{15} = \frac{4 \times 2}{5 \times 3} = \frac{4}{5} \times \frac{2}{3}$$

So $\frac{4}{5}$ of $\frac{2}{3} = \frac{4}{5} \times \frac{2}{3}$.

6. On the number line below, the measure of $\overline{AB}$ is $\frac{1}{2}$.



Find a segment with endpoint A whose measure is:

a. $\frac{1}{2}$ of $\frac{1}{2}$ $(\overline{AC})$ $\frac{1}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{1}{4})}$

b. $\frac{2}{2}$ of $\frac{1}{2}$ $(\overline{AB})$ $\frac{2}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{2}{4} \text{ or } \frac{1}{2})}$

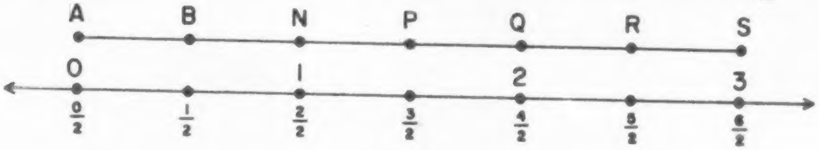
c. $\frac{3}{2}$ of $\frac{1}{2}$ $(\overline{AD})$ $\frac{3}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{3}{4})}$

d. $\frac{4}{2}$ of $\frac{1}{2}$ $(\overline{AE})$ $\frac{4}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{4}{4} \text{ or } 1)}$

e. $\frac{5}{2}$ of $\frac{1}{2}$ $(\overline{AF})$ $\frac{5}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{5}{4} \text{ or } 1\frac{1}{4})}$

f. $\frac{6}{2}$ of $\frac{1}{2}$ $(\overline{AG})$ $\frac{6}{2}$ of $\frac{1}{2}$ = $\underline{(\frac{6}{4} \text{ or } 1\frac{1}{2})}$

7. On the number line below, the measure of $\overline{AB}$ is $\frac{1}{2}$.



a. Find a segment with endpoint A whose measure is

$\frac{1}{2} + \frac{1}{2}$. $(\overline{AN})$ $\frac{1}{2} + \frac{1}{2} = \underline{(\frac{2}{2} \text{ or } 1)}$.

b. Find a segment with endpoint A whose measure is

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$. $(\overline{AP})$ $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \underline{(\frac{3}{2} \text{ or } 1\frac{1}{2})}$.

c. Find a segment with endpoint A whose measure is

$3 \times \frac{1}{2}$. $(\overline{AP})$ $3 \times \frac{1}{2} = \underline{(\frac{3}{2} \text{ or } 1\frac{1}{2})}$.

Your answer to Exercise 6f should show that $\frac{6}{2}$ of $\frac{1}{2} = \frac{6}{4}$.

Your answer to Exercise 7c should show that $3 \times \frac{1}{2} = \frac{3}{2}$.

Are $\frac{6}{4}$ and $\frac{3}{2}$ names for the same number? (Yes)

Is this true? $\frac{6}{2}$ of $\frac{1}{2} = 3 \times \frac{1}{2}$? (Yes)

P92

Exercise Set 15

1. Draw number lines and segments to show

a. $\frac{1}{3}$ of $\frac{1}{2}$

b. $\frac{3}{4}$ of $\frac{2}{5}$

c. $\frac{3}{2}$ of $\frac{7}{5}$

d. $\frac{3}{4}$ of 10

e. $\frac{6}{2}$ of $\frac{1}{4}$

2. Find by using $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

a. $\frac{1}{3} \times \frac{1}{2}$ $\left(\frac{1}{6}\right)$

b. $\frac{3}{4} \times \frac{2}{5}$ $\left(\frac{6}{20}\right)$

c. $\frac{3}{2} \times \frac{7}{5}$ $\left(\frac{21}{10}\right)$

d. $\frac{3}{4} \times 10$ $\left(\frac{30}{4}\right)$

e. $\frac{6}{2} \times \frac{1}{4}$ $\left(\frac{6}{8}\right)$

3. From your results in Exercises 1 - 2, is it true that

$\frac{1}{3}$ of $\frac{1}{2} = \frac{1}{3} \times \frac{1}{2}$? (Yes) $\frac{3}{4}$ of $\frac{2}{5} = \frac{3}{4} \times \frac{2}{5}$? (Yes) $\frac{3}{2}$ of $\frac{7}{5} = \frac{3}{2} \times \frac{7}{5}$? (Yes)

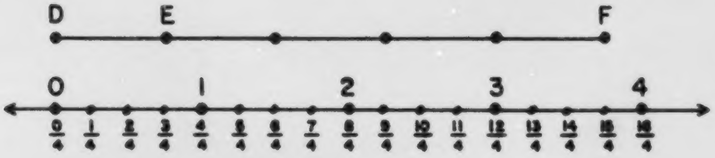
Look at the number lines and segments in Exercises 4 - 7.

4. In the diagram below, find a segment whose measure is:

a. $\frac{3}{4}$ $(\overline{DE})$

b. $5 \times \frac{3}{4}$ $(\overline{DF})$

c. $\frac{15}{4}$ $(\overline{DF})$

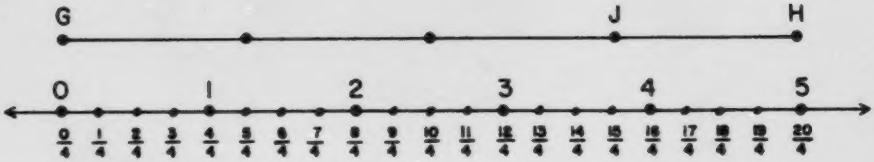


5. In the diagram below, find a segment whose measure is:

a. 5 $(\overline{GH})$

b. $\frac{3}{4} \times 5$ $(\overline{GJ})$

c. $\frac{15}{4}$ $(\overline{GJ})$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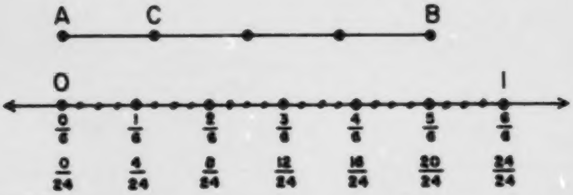
19/4
20/4

6. Find a segment whose measure is:

a. $\frac{5}{6}$ $(\overline{AB})$

b. $\frac{1}{4} \times \frac{5}{6}$ $(\overline{AC})$

c. $\frac{5}{24}$ $(\overline{AC})$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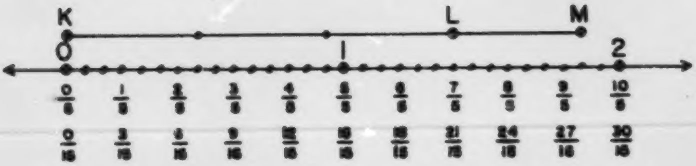
7. Find a segment whose measure is:

a. $\frac{7}{5}$ $(\overline{KL})$

b. $\frac{4}{3} \times \frac{7}{5}$ $(\overline{KM})$

c. $\frac{28}{15}$ $(\overline{KM})$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P94

Make a number line diagram to represent each problem. Then write a mathematical sentence to represent the problem and answer the question in a complete sentence.

8. The Scouts hiked from the school to a camp $3\frac{1}{2}$ miles away. They stopped to rest when they had gone $\frac{1}{3}$ of the way. How far had they walked when they stopped to rest? ($\frac{1}{3} \times 3\frac{1}{2} = m$ They had gone $1\frac{1}{6}$ miles)

9. Jane had $\frac{3}{4}$ yard of ribbon for badges. She made 4 badges of equal length. How long was each badge? ($\frac{1}{4} \times \frac{3}{4} = t$ Each badge was $\frac{3}{16}$ of a yard long.)

10. A 5 story building is 48 feet high. If the stories are of equal height, how far above the ground is the ceiling of the third story? ($\frac{3}{5} \times 48 = p$ The ceiling was $28\frac{4}{5}$ feet above ground.)

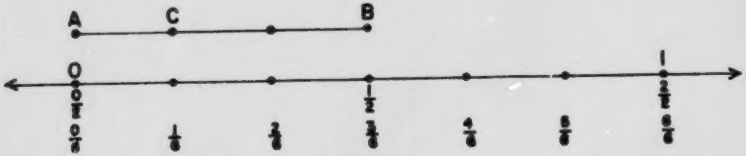
11. Bill put up shelves in his room, each $2\frac{1}{4}$ feet long. How long a board did he use for 3 shelves? ($3 \times 2\frac{1}{4} = t$ He used $6\frac{3}{4}$ feet.)

12. Sue used $\frac{3}{4}$ of a piece of toweling to make a place mat. If the piece was $\frac{2}{3}$ yard long, how long a piece did she use for the place mat? ($\frac{2}{3} \times \frac{3}{4} = r$ The piece was $\frac{1}{2}$ yard long.)

Answers, Exercise Set 15

1\.

a)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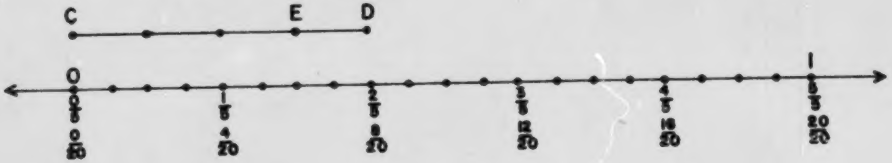
$m\overline{AC} = \frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$

b)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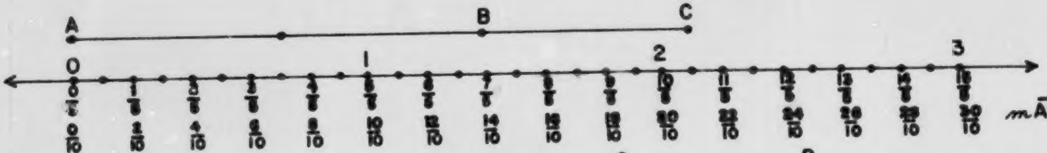
$m\overline{CD} = \frac{2}{5}$

$m\overline{CE} = \frac{3}{4}$ of $\frac{2}{5} = \frac{6}{20}$

c)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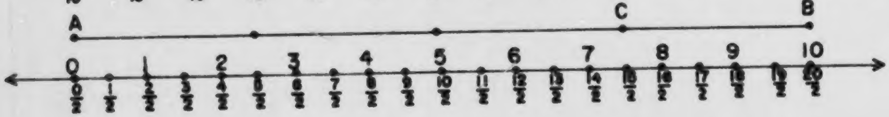
28/10
30/10

$m\overline{AB} = \frac{7}{5}$

$m\overline{AC} = \frac{3}{2}$ of $\frac{7}{5} = \frac{21}{10}$ a $\frac{1}{10}$

d)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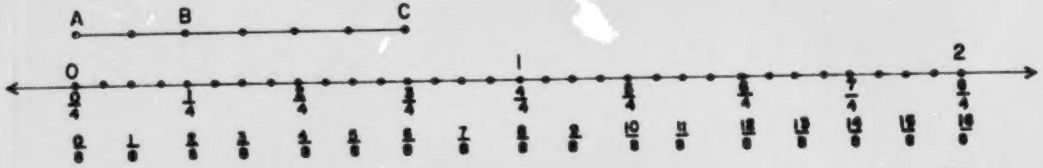
$\overline{AB} = 10$

$\overline{AC} = \frac{3}{4}$ of $10 = \frac{15}{2}$ a $\frac{1}{2}$

e)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$m\overline{AB} = \frac{1}{4}$

$m\overline{AC} = \frac{6}{2}$ of $\frac{1}{4} = \frac{6}{8}$

USING MATHEMATICAL SENTENCES TO DESCRIBE PROBLEMS

Objectives: To assist pupils in writing mathematical sentences for problems requiring more than one operation for solution.

To direct attention to certain parts of mathematical sentences and consider their relation to the problem situation.

To emphasize the use of the number line as an aid in writing mathematical sentences.

Suggested Teaching Procedure:

Have the pupils read the Exploration. Then reread the second problem and raise questions about the sentences suggested, such as the following:

In the second sentence, $(7 - 3\frac{2}{3}) + 2\frac{1}{6} = n$, what does the numeral $(7 - 3\frac{2}{3})$ represent? (Number of yards left after the first costume is made.)

In the third sentence, $7 - (3\frac{2}{3} + 2\frac{1}{6}) = n$, what does $(3\frac{2}{3} + 2\frac{1}{6})$ represent? (Number of yards used for both costumes.)

In the fifth sentence, $n + (3\frac{2}{3} + 2\frac{1}{6}) = 7$, what does the whole left side of the equation represent? (Sum of number of yards left and number of yards used.)

Call attention to the use of the number line for representing the relation between the numbers in the problem.

Discuss the third problem in a similar manner, and illustrate it by using the number line.

Propose some mathematical sentences and ask the pupils to make up some problems they represent.

Samples: $100 - (52 + 26) = n$. (Joe bought eggs costing 52 cents and bread costing 26 cents. How much change should he get from $1.00?)

$80 + (n \times 30) = 350$. (Jane has 80 cents. If she can save 30 cents a week, in how many weeks will she have $3.50?

USING MATHEMATICAL SENTENCES TO DESCRIBE PROBLEMS

Exploration

1. Richard's new foreign car travels 29 miles on one gallon of gas. How many miles will it travel on 7 gallons?

$$29 \times 7 = t$$

$$203 = t$$

Richard's car will go 203 miles on 7 gallons of gas. What relationship in the problem is expressed in the mathematical sentence? (Car travels 7 times as far in 7 hours as in 1 hour)

P96

2. Consider the problem: Mary's mother bought 7 yards of material to make two costumes. She used $3\frac{2}{3}$ yards for one costume and $2\frac{1}{6}$ yards for the other. How many yards of material did she have left?

To solve this problem, what question should you ask first? Should you ask, "How many yards did she use in all for the two costumes?"

Suppose you call this number k. Does

$$3\frac{2}{3} + 2\frac{1}{6} = k \ ?$$

What question could you answer next?

Suppose n is the number of yards left. Does $7 - k = n$? Which of these mathematical sentences is a correct representation for the problem?

$7 - 3\frac{2}{3} + 2\frac{1}{6} = n$ (wrong)

$(7 - 3\frac{2}{3}) + 2\frac{1}{6} = n$ (wrong)

$7 - (3\frac{2}{3} + 2\frac{1}{6}) = n$ (correct)

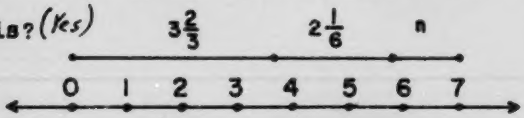
What about these sentences? Would any of them be correct also?

$(7 - 3\frac{2}{3}) - 2\frac{1}{6} = n$ (correct)

$n + (3\frac{2}{3} + 2\frac{1}{6}) = 7$ (correct)

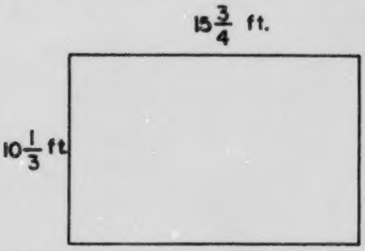
$3\frac{2}{3} + 2\frac{1}{6} = 7 - n$ (correct)

Can this problem be represented by segments on the number line like this? (Yes)



P97

3. Below is a diagram of Steve's garden. He wants to put a wire fence around it. The wire costs $13\frac{1}{2}$ cents a foot. What will be the cost of the wire?



Do you have to know how much wire is needed before you can answer the question, "What will be the cost of the wire?" (Yes)

Let m represent the perimeter of the garden in feet.

Does $(10\frac{1}{3} \times 2) + (15\frac{3}{4} \times 2) = m$? (Yes)

What question could you answer next?

Let n = the number of cents the wire costs.

Does $13\frac{1}{2} \times m = n$? (Yes)

Which of the following mathematical sentences best expresses the relationship in the problem?

$15\frac{3}{4} \times [(2 + 10\frac{1}{3}) \times (2 \times 13\frac{1}{2})] = n$ (wrong)

$(15\frac{3}{4} \times 2) + 10\frac{1}{3} \times (2 \times 13\frac{1}{2}) = n$ (wrong)

$[(15\frac{3}{4} \times 2) + (10\frac{1}{3} \times 2)] \times 13\frac{1}{2} = n$ (wrong)

Notice symbols [ ] called brackets are used to group parentheses that name only one number.

What about these sentences? Would any of them be correct also?

$[(15\frac{3}{4} + 15\frac{3}{4}) + (10\frac{1}{3} + 10\frac{1}{3})] \times 13\frac{1}{2} = n$ (correct)

$[(15\frac{3}{4} + 15\frac{3}{4}) \times 13\frac{1}{2}] + [(10\frac{1}{3} + 10\frac{1}{3}) \times 13\frac{1}{2}] = n$ (correct)

$(15\frac{3}{4} + 15\frac{3}{4} + 10\frac{1}{3} + 10\frac{1}{3}) \times 13\frac{1}{2} = n$ (correct)

P98

Exercise Set 16

Read the following carefully, write the relationship in each problem as a mathematical sentence, solve, and answer the question asked in the problem:

1. A recipe calls for $\frac{1}{4}$ cup butter. If you make only $\frac{1}{2}$ of the recipe, how much butter do you need? ($\frac{1}{2} \times \frac{1}{4} = p$ You need $\frac{1}{8}$ cup of butter.)

2. John lives $\frac{1}{3}$ mile from school. Harry lives only half that distance from school. How far from the school does Harry live? ($\frac{1}{2} \times \frac{1}{3} = n$ Harry lives $\frac{1}{6}$ mile from school.)

3. Mrs. Morgan bought half a cake for her family of four. If she made all servings the same size, what part of a cake was each person served? ($\frac{1}{4} \times \frac{1}{2} = w$ Each person was served $\frac{1}{8}$ cake.)

Exercise Set 17

1. Peter bought $\frac{1}{2}$ pound of cheese for school lunches. The first day he used $\frac{1}{5}$ of the amount he bought. How much cheese did he use? ($\frac{1}{5} \times \frac{1}{2} = r$ He used $\frac{1}{10}$ pound of cheese.)

2. Sara is supposed to practice the piano $\frac{1}{2}$ hour each day. She practiced only $\frac{1}{3}$ of that time on Saturday. What part of an hour did she practice on Saturday? ($\frac{1}{3} \times \frac{1}{2} = q$ Sara practiced $\frac{1}{6}$ hour on Saturday.)

3. Terry ate $\frac{1}{3}$ dozen cookies after lunch. He ate $\frac{1}{4}$ dozen cookies after dinner. What part of a dozen cookies did he eat? ($\frac{1}{3} + \frac{1}{4} = n$ Terry ate $\frac{7}{12}$ dozen cookies.)

4. The distance from the library to the city hall is $\frac{3}{4}$ of a mile. What part of a mile will you have gone if you walk $\frac{3}{4}$ of this distance? ($\frac{3}{4} \times \frac{3}{4} = d$ You will have walked $\frac{9}{16}$ mile.)

5. Ned needs a piece of canvas $8\frac{1}{4}$ feet long. He has one piece $2\frac{3}{4}$ feet long and another piece $3\frac{1}{2}$ feet long. How much does he still need? ($8\frac{1}{4} - (3\frac{1}{2} + 2\frac{3}{4}) = t$ He will need 2 ft. of canvas.)

6. One-half the pupils of a school are going to a concert. These children will be taken in 5 buses. What part of the pupils of the school will ride in each bus? ($\frac{1}{5} \times \frac{1}{2} = m$ $\frac{1}{10}$ of the pupils of the school will ride in each bus.)

7. A gallon of cream weighs 8.4 pounds, a gallon of milk weighs 8.6 pounds, and a gallon of water weighs 8.3 pounds. How many pounds will a gallon of cream, a gallon of milk, and a gallon of water weigh together? ($8.4 + 8.6 + 8.3 = t$ A gallon of cream, a gallon of milk, and a gallon of water weigh 25.3 pounds together.)

8. Carol feeds her dog $\frac{3}{4}$ pounds of meat daily. She feeds him twice a day. What part of a pound of meat does the dog get at each meal? ($\frac{1}{2} \times \frac{3}{4} = s$ The dog eats $\frac{3}{8}$ pounds of meat at each meal.)

9. The record speed for an airplane in 1960 was 1,526 miles per hour. The record speed for an automobile was 394.19 miles per hour. How much greater was the speed recorded for an airplane? ($1526 - 394.19 = w$ The speed of the plane is 1131.81 miles per hour greater.)

10. Mary weighs $62\frac{1}{2}$ pounds. Her brother weighs $\frac{1}{2}$ as much as she weighs. What does he weigh? ($\frac{1}{2} \times 62\frac{1}{2} = n$ Mary's brother weighs $31\frac{1}{4}$ pounds.)

11. George lives 2.7 miles from school. He makes one round trip each day. How many miles does he walk to school each week? ($2.7 \times 5 = q$ George walks 13.5 miles to school each week.)

12. Mrs. Marks bought a $5\frac{1}{2}$ pound roast and $\frac{3}{4}$ pound of ground meat. How much meat did she buy altogether?
($5\frac{1}{2} + \frac{3}{4} = m$ Mrs. Marks bought $6\frac{1}{4}$ pounds of meat.)

13. Mr. Hayes drove 42.3 miles per hour for 3 hours. How many miles did he drive? ($42.3 \times 3 = r$ Mr. Hayes drove 126.9 miles.)

14. Mike's garden is $15\frac{1}{2}$ feet by $20\frac{1}{2}$ feet. What is the area of the garden? ($15\frac{1}{2} \times 20\frac{1}{2} = y$ Mike's garden is $317\frac{3}{4}$ sq. ft.

15. Tim's dog eats $\frac{1}{4}$ pound of food in the morning and $\frac{1}{3}$ pound of food in the afternoon. How much food does Tim's dog eat during one week? ($7 \times (\frac{1}{4} + \frac{1}{3}) = n$ Tim's dog eats $4\frac{1}{12}$ pounds in one week.)

RATIONAL NUMBERS WITH SETS OF OBJECTS

Objectives: To review ways of picturing products of rational numbers using regions and segments.

To review use of a rational number to describe part of a set of objects.

To give added meaning to multiplication of rational numbers by associating them with sets of objects.

To show that finding such numbers as $\frac{3}{4}$ of the number of objects in a set ($\frac{1}{2}$ of 4, $\frac{3}{4}$ of 12, $\frac{5}{8}$ of 16, etc.) is an operation which gives the same answer as multiplication.

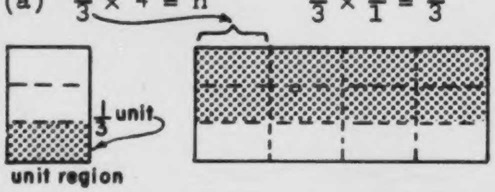
Materials: Collections of objects which can be separated to show subsets; arrays suggested on page 89 number line.

Vocabulary: Set, subset, array.

Suggested Teaching Procedure :

Recall how rectangular regions and line segments have been used to give meaning to products of rational numbers such as $\frac{2}{3} \times 4$; $\frac{3}{4} \times \frac{1}{2}$; $3 \times \frac{3}{8}$, etc. You may wish to have children illustrate how these different physical situations give meaning to these products, as,

(a) $\frac{2}{3} \times 4 = n$ $\qquad \frac{2}{3} \times \frac{4}{1} = \frac{8}{3}$



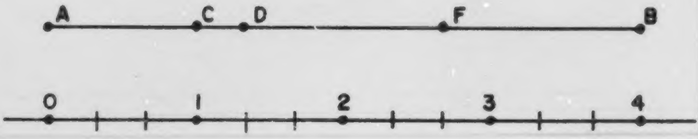
$m\overline{AB} = 4$

$m\overline{AC} = 1$

$m\overline{AD} = \frac{1}{3} \times 4$

$m\overline{AF} = \frac{2}{3} \times 4$

$m\overline{AF} = \frac{8}{3}$



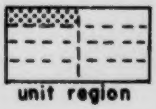
(b) $\frac{3}{4} \times \frac{1}{2} = n$ $\qquad$ $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$

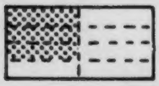
$m\overline{AB} = 1$

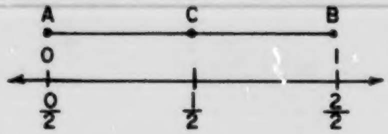
$m\overline{AC} = \frac{1}{2}$

$m\overline{AD} = \frac{3}{4} \times \frac{1}{2}$

$m\overline{AD} = \frac{3}{8}$


unit region




A
C
B
0
1
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$

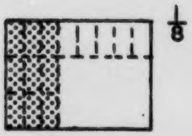
(c) $3 \times \frac{3}{8} = n$

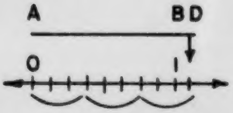
$m\overline{AB} = 1$

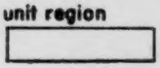
$m\overline{AC} = \frac{3}{8}$

$m\overline{AD} = 3 \times \frac{3}{8}$

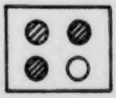
$m\overline{AD} = \frac{9}{8}$



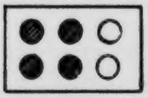



unit region

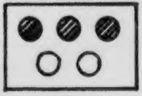
Then recall how, in earlier grades, we gave meaning to rational numbers by such models as subsets of objects (shaded) compared with the set of objects, as shown here:



$\frac{3}{4}$

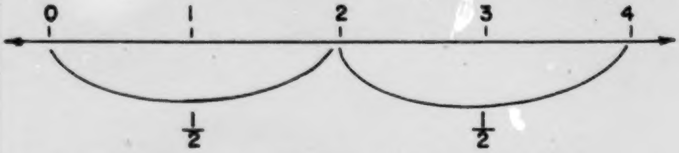


$\frac{4}{6}$



$\frac{3}{5}$

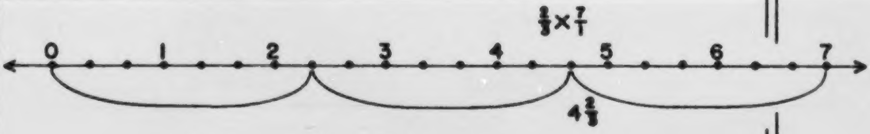
Then suggest that we think again of a set of four objects. Arrange on a number line. Find $\frac{1}{2}$ of the objects--as we did in $(\frac{2}{3} \times 4)$.


0
1
2
3
4
$\frac{1}{2}$
$\frac{1}{2}$

We can write the mathematical sentence $\frac{1}{2}$ of $\frac{4}{1} = \frac{4}{2} = 2$.

Use some other similar examples, but be certain to include one like:

> What is $\frac{2}{3}$ of 7 objects? What mathematical sentence can be written to represent this?



$$\frac{2}{3}\times\frac{7}{1}=\frac{14}{3}=4\frac{2}{3}$$

Of course, in application, if objects are not congruent and cannot be separated into congruent parts, rational numbers should not be used. Instead, the mathematical sentence $p = (n \times q) + r$ is appropriate.

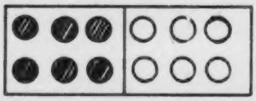
(Continue with the exploration in the pupil text.)

The following development is included if teacher wishes to extend the discussion to finding $\frac{1}{2}$ of $\frac{3}{4}$ of 12. It is enrichment material and its use is optional.

Take a collection of congruent objects (e.g. a set of 12 objects) such that each object can be separated into congruent parts. Arrange in a series of arrays such as suggested below. For each arrangement suggested questions and comments are given.

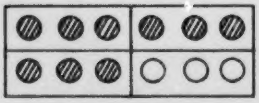
(1) Using a 2 by 6 array: Two problems using a 2 by 6 array.

(a)

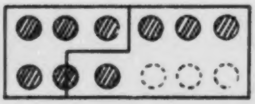


Find $\frac{1}{2}$ of 12 objects. Separate into 2 equivalent sets. We can write: $\frac{1}{2}\times\frac{12}{1}=n$.

(b) No. 1



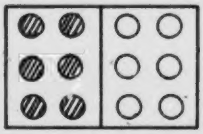
(b) No. 2



Find $\frac{1}{2}$ of $\frac{3}{4}$ of 12 objects. First find $\frac{3}{4}$ of 12 objects (No. 1). We can write: $\frac{3}{4} \times \frac{12}{1} = n$. Now using this same set of objects find $\frac{1}{2}$ of the set of 9 objects. (See No. 2). Now we can write the complete sentence: $\frac{1}{2} \times (\frac{3}{4} \times \frac{12}{1}) = n$.

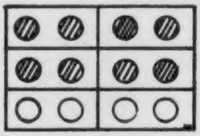
(2) Using a 3 by 4 array: Two suggested problems using a 3 by 4 array.

(a)



Find $\frac{1}{2}$ of 12 objects. The mathematical sentence: $\frac{1}{2} \times \frac{12}{1} = n$.

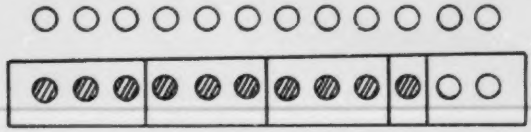
(b)



Find $\frac{2}{3}$ of $\frac{4}{6}$ of 12 objects. First find $\frac{4}{6}$ of 12 objects, then find $\frac{2}{3}$ of 8 objects, [or $(\frac{4}{6} \times \frac{12}{1})$ objects]. See that each $\frac{1}{3}$ of 8 is $(\frac{2}{4} + \frac{2}{3})$. Then, $\frac{2}{3}$ of 8 is $(4 + \frac{4}{3})$ or $5\frac{1}{3}$. The mathematical sentence: $\frac{2}{3} \times (\frac{4}{6} \times \frac{12}{1}) = n$.

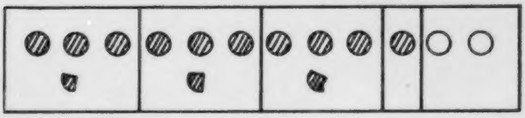
(3) Using a 1 by 12 array. One suggested problem using a 1 by 12 array also, with association with number line.

(a)



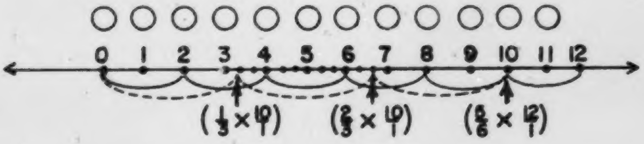
Find $\frac{1}{3}$ of $\frac{5}{6}$ of 12 objects. First find $\frac{5}{6}$ of 12 objects. Then find $\frac{1}{3}$ of this set. See that we must separate 1 object into 3 component parts.

(b)



$$\frac{1}{3} \times \left(\frac{5}{6} \times \frac{12}{1}\right) = \frac{1}{3} \times \frac{60}{6}$$
$$= \frac{1}{3} \times \frac{10}{1} = \frac{10}{3} = 3\frac{1}{3}$$

Also associate this with the number line.


0
1
2
3
4
5
6
7
8
9
10
11
12
$(\frac{1}{3} \times \frac{10}{1})$
$(\frac{2}{3} \times \frac{10}{1})$
$(\frac{5}{6} \times \frac{12}{1})$

The dotted line shows $(\frac{5}{6} \times \frac{12}{1})$ separated into 3 congruent line segments.

We can write: $\frac{1}{3} \times \left(\frac{5}{6} \times \frac{12}{1}\right) = \frac{1}{3} \times \frac{10}{1}$

$$\frac{1}{3} \times \frac{10}{1} = \frac{10}{3} = 3\frac{1}{3}$$

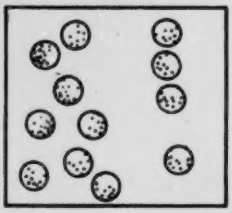
RATIONAL NUMBERS WITH SETS OF OBJECTS

You have illustrated products of rational numbers by using rectangular regions and by using segments on the number line.

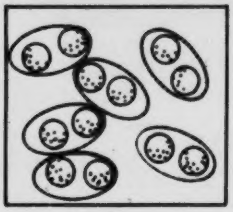
Multiplication of rational numbers can also be used to answer some kinds of questions about sets of objects.

Exploration

Picture A represents a set of golf balls. In picture B this set is separated into subsets, with the same number of balls in each subset.



A

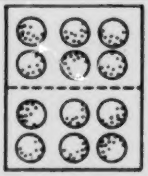


B

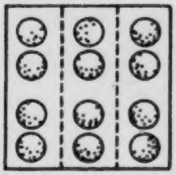
1. How many balls are in the set? (12) What rational number represents the part of the set in each subset? ($\frac{1}{6}$) How many balls are in each subset? (2) Does the sentence, $\frac{1}{6}$ of 12 balls are 2 balls, describe this situation? (Yes)

2. Draw pictures of the set of balls separated into fewer subsets, with the same number of balls in each subset. Can you do this in more than one way? (Answers will vary) (4 subsets, 3 subsets, 2 subsets)

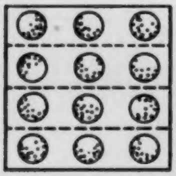
3. **Pictures C, D, E, F, G show the balls arranged in arrays. The broken lines in each picture show the separation of the set into subsets, with the same number of balls in each subset.**



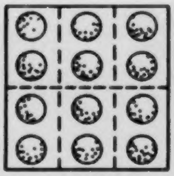
C



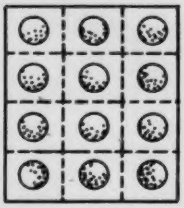
D



E



F



G

4. **Write a sentence like the one in Exercise 1 which describes each picture.** (C. $\frac{1}{2}$ of $12 = 6$ D. $\frac{1}{3}$ of $12 = 4$ E. $\frac{1}{4}$ of $12 = 3$ F. $\frac{1}{6}$ of $12 = 2$ G. $\frac{1}{12}$ of $12 = 1$)

5. **Use the pictures to find**

a. $\frac{2}{3}$ **of 12 balls** (8) d. $\frac{1}{2}$ **of 12 balls** (6)

b. $\frac{4}{6}$ **of 12 balls** (8) e. $\frac{2}{4}$ **of 12 balls** (6)

c. $\frac{5}{12}$ **of 12 balls** (5) f. $\frac{6}{12}$ **of 12 balls** (6)

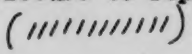
6. a. Is $\frac{2}{3}$ of 12 the same number of balls as $\frac{2}{3} \times \frac{12}{1}$? (Yes)

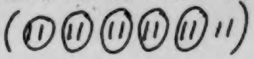
   b. Is $\frac{7}{12}$ of 12 the same number as $\frac{7}{12} \times \frac{12}{1}$? (Yes)

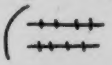
   c. Is $\frac{5}{6}$ of 12 the same number as $\frac{5}{6} \times \frac{12}{1}$? (Yes)

   d. Can you use multiplication of rational numbers to find the number of objects in a part of a set? (Yes)

7. Do your answers in Exercise 6 agree with what you found to be true when you used segments on the number line?

8. a. Draw a picture to represent a set of 12 licorice sticks. 

   b. You wish to divide the licorice sticks among 5 boys. Draw rings around subsets to show the whole sticks each boy will get. 

   c. Show on your drawing how you will divide up the remaining sticks.  Each boy will get $\frac{2}{5}$)

   d. Write the rational number which represents the number of sticks each boy will get. Is your result equal to $\frac{1}{5} \times \frac{12}{1}$? (Yes)

Exercise Set 18

1. Draw pictures of golf balls. Then write the mathematical sentences.

   a. $\frac{4}{5}$ of 15 $\left(\frac{4}{5} \times \frac{15}{1} = n;\ n = 12\right)$

   b. $\frac{7}{8}$ of 24 $\left(\frac{7}{8} \times \frac{24}{1} = r;\ r = 21\right)$

   c. $\frac{9}{10}$ of 30 $\left(\frac{9}{10} \times \frac{30}{1} = t;\ t = 27\right)$

   d. $\frac{3}{7}$ of 28 $\left(\frac{3}{7} \times \frac{28}{1} = r;\ r = 12\right)$

   e. $\frac{7}{5}$ of 20 $\left(\frac{7}{5} \times \frac{20}{1} = p;\ p = 28\right)$

   f. $\frac{5}{4}$ of 24 $\left(\frac{5}{4} \times \frac{24}{1} = t;\ t = 30\right)$

2. Find n:

a. $\frac{4}{5} \times \frac{15}{1} = n\left(\frac{60}{5}\right)$ or (12)

b. $\frac{7}{8} \times \frac{24}{1} = n\left(\frac{168}{8}\right)$ or (21)

c. $\frac{9}{10} \times \frac{30}{1} = n\left(\frac{270}{10}\right)$ or (27)

d. $\frac{3}{7} \times \frac{28}{1} = n\left(\frac{84}{7}\right)$ or (12)

e. $\frac{7}{5} \times \frac{20}{1} = n\left(\frac{140}{5}\right)$ or (28)

f. $\frac{5}{4} \times \frac{24}{1} = n\left(\frac{120}{4}\right)$ or (30)

3. Compare your results in Exercises 1 and 2.

4. Only two-thirds of the pupils in Bedford School can be seated in the auditorium, so only two-thirds of the classes may go to the assembly. There are 33 classes. How many classes may go? ($\frac{2}{3} \times 33 = t$. 22 classes may go.)

5. Jim said that $\frac{9}{10}$ of his 40 tomato plants had tomatoes on them. How many plants had tomatoes? How many did not? ($\frac{9}{10} \times 40 = r$; $40 - r = t$ 36 plants had tomatoes. Four plants had no tomatoes)

6. The refreshment committee estimated that $\frac{3}{4}$ of the pupils and parents would come to the class picnic. If there were 84 pupils and parents, how many people did the committee think would come? ($\frac{3}{4}$ of $84 = n$. The committee thought 63 people would come.)

7. A class had a supply of 1 gross of pencils (144) when school began. A month later they had 84 left. What rational number tells what part of their supply they had left? $\left(\frac{7}{12}\right)$

8. Jane had 24 questions right on a test with 30 questions. What rational number tells what part of the test she answered correctly? $\left(\frac{24}{30}\right)$

PROPERTIES OF RATIONAL NUMBERS (OVERVIEW)

In these three sections we inquire whether the familiar properties of the operation of multiplication of whole numbers apply when the factors are rational numbers, and whether the rational numbers have any new properties which the whole numbers do not have.

In the first of the three sections (Properties of Rational Numbers), the Commutative and Associative, and the Properties of Zero and One are verified for multiplication of rational numbers. The Distributive Property is also verified.

One purpose of the second section (Problems about Travel) is to call attention to a property which pupils often associate with multiplication, but which is actually true only when both factors are larger than 1. This has to do with the relation of size of product to size of factors. Pupils frequently generalize, on the basis of experience with whole numbers, that "when you multiply you get a bigger number." This tends to confuse them when they multiply rational numbers and the generalization does not necessarily hold.

In the third section (A New Property of Rational Numbers), the Reciprocal Property is developed. If the product of two numbers is 1, each number is the reciprocal of the other. While the fraction numeral provides an easy clue as to what the reciprocal of a given number must be, it is important to note that the Reciprocal Property applies to the rational number, not to the fraction numeral. For example, the number named by the fraction $\frac{4}{3}$ has as reciprocal the number named by $\frac{3}{4}$, or $\frac{75}{100}$, or 0.75, etc. The Reciprocal Property is important in itself, because it will be used in the development of the operation of division of rational numbers. It is also important as an instance of a property possessed by the set of rational numbers which was not possessed by the set of whole numbers. For example, there is no whole number u such that $8 \times u = 1$.

PROPERTIES OF RATIONAL NUMBERS

Objective:	To verify the Commutative, Associative, Distributive Properties and the Properties of Zero and One for the rational numbers.
Materials:	Number line, rectangular regions.
Vocabulary:	Commutative, associative, distributive, closure, properties of zero and one.

Suggested Teaching Procedure:

You know several properties of multiplication of whole numbers. Can you give examples of some of these properties? (Pupils should suggest such illustrations as $3 \times 4 = 4 \times 3$, $2 \times (3 \times 4) = (2 \times 3) \times 4$, 2×3 is a whole number, $0 \times 5 = 0$, $1 \times 5 = 1$. Also, $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$, which relates multiplication and addition.) What are the names of these properties? (Commutative, Associative, Closure, Zero, One, Distributive.)

Do you think that multiplication of rational numbers has the same properties? The exercises on these pages suggest ways of finding out.

> Have the pupils do the exercises in the Exploration and tell how the results suggest that the properties do hold for rational numbers.
>
> Summarize by developing a list (with illustrations) of the properties.

PROPERTIES OF RATIONAL NUMBERS

Exploration

You know several properties of the operation of multiplication with whole numbers: Commutative, Associative, Closure, Property of Zero, and Property of One. You also know the Distributive Property, which relates multiplication and addition of whole numbers.

1. Illustrate each of these properties with whole numbers. (Answers will vary)
   Make diagrams (using rectangular regions or segments on the number line) to illustrate the following products:

2. a. $\frac{2}{3} \times \frac{5}{4}$ (Answers will vary)

   b. $\frac{5}{4} \times \frac{2}{3}$

3. a. $\frac{2}{5} \times 3$ (Answers will vary)

   b. $3 \times \frac{2}{5}$

4. Find the products in Exercise 2 and Exercise 3 using $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. (2a $\frac{10}{12}$ 2b. $\frac{10}{12}$ 3a. $\frac{6}{5}$ 3b. $\frac{6}{5}$)

5. a. Is $\frac{2}{3} \times \frac{5}{4} = \frac{5}{4} \times \frac{2}{3}$ a true statement? (Yes)

   b. Is $\frac{2}{5} \times 3 = 3 \times \frac{2}{5}$ a true statement? (Yes)

6. What property for multiplication of rational numbers is suggested by Exercises 2 - 5 ? (Commutative)

Find s and t:

7. a. $s = (\frac{1}{4} \times \frac{2}{3}) \times \frac{2}{5}$ $\left(\frac{4}{60}\right)$

   b. $t = \frac{1}{4} \times (\frac{2}{3} \times \frac{2}{5})$ $\left(\frac{4}{60}\right)$

8. a. $s = (7 \times \frac{3}{4}) \times \frac{5}{6}$ $\left(\frac{105}{24}\right)$

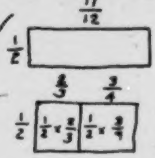
   b. $t = 7 \times (\frac{3}{4} \times \frac{5}{6})$ $\left(\frac{105}{24}\right)$

9. a. What is true of the products s and t in Exercise 7? (They are equal.)

   b. What is true of the products s and t in Exercise 8? (They are equal.)

10. What property of multiplication of rational numbers is suggested by Exercises 7 - 8? (Associative)

11. Find the numbers n and t by using drawings like the ones on Page 70 for Exercise 4.



$\frac{1}{2} \times \frac{17}{12} = \frac{17}{24} = n$

$\frac{1}{2} \times \frac{2}{3} = \frac{2}{6} = \frac{8}{24}$

$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8} = \frac{9}{24}$

$\frac{8}{24} + \frac{9}{24} = \frac{17}{24} = t$

   a. $n = \frac{1}{2} \times (\frac{2}{3} + \frac{3}{4})$

   b. $t = (\frac{1}{2} \times \frac{2}{3}) + (\frac{1}{2} \times \frac{3}{4})$

12. Which of these sentences is true in Exercise 11?

    $n > t$ $n = t$ $n < t$

    $(n = t)$

13. State a property of rational numbers suggested by Exercises 11 - 12. (Distributive)

Find r and s:

14. a. $\frac{0}{8} \times \frac{3}{4} = r$ $\left(\frac{0}{32} = 0\right)$

    b. $\frac{5}{6} \times \frac{0}{3} = s$ $\left(\frac{0}{18} = 0\right)$

15. What property of rational numbers does Exercise 14 suggest? (Multiplicative Property of Zero.)

16. Find r and s. Express your answer in simplest form.

a. $r = \frac{3}{3} \times \frac{3}{5}$ $(\frac{9}{15} = \frac{3}{5}$ b. $s = \frac{7}{3} \times \frac{10}{10}$ $(\frac{70}{30} = \frac{7}{3}$

17. Compare each product in Exercise 16 with the factors. What do you observe? (Product equals one factor.)

18. State a property of rational numbers suggested by Exercise 16. (Multiplication Property of One.)

19. You are used to working with rational numbers that are easy to picture. Here are some names for less commonly used rational numbers: $\frac{178}{249}$ and $\frac{31}{155}$.

a. Can you imagine a rectangular region whose sides have these measures?

b. How would you find the measure of the region? $(\frac{178}{249} \times \frac{31}{155})$

c. What kind of number would your result be? (Rational)

d. Make up two other strange rational numbers. (Answers will vary) If they were measures of the sides of a rectangular region, what kind of number would the measure of the region be? (Rational)

e. What property of multiplication of rational numbers does this suggest? (Closure: The product of any two rational numbers is a rational number)

Exercise Set 19

Copy and complete these multiplication charts. Express products in simplest form.

1.

x	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{8}{15}$
$\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{3}{5}$
$\frac{4}{5}$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{16}{25}$

2.

x	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{1}{2}$
$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{2}{3}$
$\frac{3}{4}$	0	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{3}{4}$
1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1

3.

x	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	
2	$\frac{2}{2}$ or 1	$1\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{3}{5}$	
3	$1\frac{1}{2}$	2	$2\frac{1}{4}$	$2\frac{2}{5}$	
4	2	$2\frac{2}{3}$	3	$3\frac{1}{5}$	
5	$2\frac{1}{2}$	$3\frac{1}{3}$	$3\frac{3}{4}$	4	

4. What property (or properties) do you find illustrated in each chart? (1. Commutative Size of Products 2. Commutative, Zero, One, Size of Products. 3 Size of Products.)

SIZE OF PRODUCTS

Objective: To call attention to the relation of size of product to size of factors. Some products may be less than either factor.

Materials: Number line.

See also notes on Page 202

Children often recognize that finding the distance travelled in 3 minutes at a given rate per minute can be done by using the operation of multiplication, but fail to see that the same operation (multiplication) applies in case the number of minutes is less than one. Hence the first line in the table has the last entry written "4 × 600 = 2400", rather than merely "2400". The same form should be continued for the other entries to emphasize that they are all instances of multiplication.

The number line is used in the Exploration to serve as a geometric model for the generalizations about size of products called for in Ex. 11.

Ex. 10 reinforces the observations about properties of rational numbers and size of products of rational numbers.

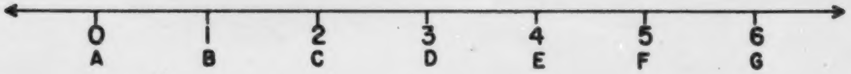
P109

SIZE OF PRODUCTS

Exploration

A bug named Willie crawls along a crack in the floor. He travels 2 feet per minute.

Suppose the crack represents a number line.



1. How far does Willie crawl in 3 minutes? (6 feet) If he starts at A, what point does he reach in 1 minute? (C) in 2 minutes? (E) in $2\frac{1}{2}$ minutes? (F) in 3 minutes? (G)

2. For each part of Exercise 1, write a mathematical sentence which shows the relation between the time Willie crawls, his speed, and the distance he goes. ($3 \times 2 = 6$; $1 \times 2 = 2$; $2 \times 2 = 4$; $2\frac{1}{2} \times 2 = 5$; $3 \times 2 = 6$)

3. Between which labeled points will Willie be when he has crawled $2\frac{3}{4}$ minutes? (E-F) $1\frac{1}{4}$ minutes? (C-D) $\frac{2}{3}$ minute? (B-C) $\frac{2}{5}$ minute? (A-B)

4. Write a mathematical sentence for each part of Exercise 3. ($2\frac{3}{4} \times 2 = 5\frac{1}{2}$. $1\frac{1}{4} \times 2 = 2\frac{1}{2}$. $\frac{2}{3} \times 2 = 1\frac{1}{3}$. $\frac{2}{5} \times 2 = \frac{4}{5}$.)

5. What operation is indicated in each mathematical sentence in Exercises 2 and 4? (Multiplication)

6. a. If Willie starts at A and crawls less than one minute, on which segment must he be? ($\overline{AC}$)

   b. If Willie starts at A and crawls more than one minute, where will he be? (Beyond C, or on ray $\overrightarrow{CD}$.)

Exercise Set 20

1. A jet plane travels 600 miles an hour. At that rate, how far does it travel in 4 hours? in 3 hours? in $2\frac{1}{2}$ hours? in $1\frac{1}{4}$ hours? 1 hour? $\frac{3}{4}$ hour? $\frac{1}{2}$ hour? $\frac{1}{3}$ hour? $\frac{1}{10}$ hour?

   Record these facts in the table below.

JET PLANE

Number of Hours	Number of Miles per Hour	Total Number of Miles Traveled
4	600	$4 \times 600 = 2400$
3	600	($3 \times 600 = 1800$)
$2\frac{1}{2}$	600	($2\frac{1}{2} \times 600 = 1500$)
$1\frac{1}{4}$	600	($1\frac{1}{4} \times 600 = 750$)
1	600	($1 \times 600 = 600$)
$\frac{3}{4}$	600	($\frac{3}{4} \times 600 = 450$)
$\frac{1}{2}$	600	($\frac{1}{2} \times 600 = 300$)
$\frac{1}{3}$	600	($\frac{1}{3} \times 600 = 200$)
$\frac{1}{10}$	600	($\frac{1}{10} \times 600 = 60$)

2. What operation did you use to answer each part of Exercise 1? (Multiplication)

3. Suppose the Scouts hike 3 miles an hour. At that rate, how far do they walk in 4 hours? In 3 hours? Make a table like the one above. Use the same numbers of hours.
   (Last Column $4 \times 3 = 12$; $3 \times 3 = 9$; $2\frac{1}{2} \times 3 = 7\frac{1}{2}$; $1\frac{1}{4} \times 3 = 3\frac{3}{4}$; $1 \times 3 = 3$; $\frac{3}{4} \times 3 = 2\frac{1}{4}$; $\frac{1}{2} \times 3 = 1\frac{1}{2}$; $\frac{1}{3} \times 3 = 1$; $\frac{1}{10} \times 3 = \frac{3}{10}$)

4. A fast, lively turtle walks $\frac{1}{8}$ mile an hour. Make a table like the one above for the turtle. What operation do you use to find his distances? (Last column: $4 \times \frac{1}{8} = \frac{1}{2}$; $3 \times \frac{1}{8} = \frac{3}{8}$; $2\frac{1}{2} \times \frac{1}{8} = \frac{5}{16}$; $1\frac{1}{4} \times \frac{1}{8} = \frac{5}{32}$; $1 \times \frac{1}{8} = \frac{1}{8}$; $\frac{3}{4} \times \frac{1}{8} = \frac{3}{32}$; $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$; $\frac{1}{3} \times \frac{1}{8} = \frac{1}{24}$; $\frac{1}{10} \times \frac{1}{8} = \frac{1}{80}$.)

Use your tables for Exercises 1, 3, and 4 to answer these questions.

5. Make these sentences true by putting >, <, or = in the blank.

	Plane	Scouts	Turtle
a.	4×600 (>) 600	4×3 (>) 3	$4 \times \frac{1}{8}$ (>) $\frac{1}{8}$
b.	3×600 (>) 600	3×3 (>) 3	$3 \times \frac{1}{8}$ (>) $\frac{1}{8}$
c.	$2\frac{1}{2} \times 600$ (>) 600	$2\frac{1}{2} \times 3$ (>) 3	$2\frac{1}{2} \times \frac{1}{8}$ (>) $\frac{1}{8}$
d.	$1\frac{1}{4} \times 600$ (>) 600	$1\frac{1}{4} \times 3$ (>) 3	$1\frac{1}{4} \times \frac{1}{8}$ (>) $\frac{1}{8}$
e.	1×600 (=) 600	1×3 (=) 3	$1 \times \frac{1}{8}$ (=) $\frac{1}{8}$
f.	$\frac{3}{4} \times 600$ (<) 600	$\frac{3}{4} \times 3$ (<) 3	$\frac{3}{4} \times \frac{1}{8}$ (<) $\frac{1}{8}$
g.	$\frac{1}{2} \times 600$ (<) 600	$\frac{1}{2} \times 3$ (<) 3	$\frac{1}{2} \times \frac{1}{8}$ (<) $\frac{1}{8}$
h.	$\frac{1}{3} \times 600$ (<) 600	$\frac{1}{3} \times 3$ (<) 3	$\frac{1}{3} \times \frac{1}{8}$ (<) $\frac{1}{8}$
i.	$\frac{1}{10} \times 600$ (<) 600	$\frac{1}{10} \times 3$ (<) 3	$\frac{1}{10} \times \frac{1}{8}$ (<) $\frac{1}{8}$

6. Look at the column in Exercise 5 about the jet plane.

   a. What factor is shown in each product expression? (600)

   b. In which lines of the table is the product greater than this factor? (a, b, c, d) equal to this factor? (e) less than this factor? (f, g, h, i)

   c. How do you explain your observation in b ? (other factor >1, =1, or <1)

7. Look at the column in Exercise 5 about the Scouts.

   a. What factor is shown in each product expression? (3)

   b. In which lines of the table is the product greater than this factor? (a, b, c, d) equal to this factor? (e) less than this factor? (f, g, h, i)

   c. How do you explain your observation in b? (Same as 6c)

8. Answer the questions in Exercise 6, using the turtle column. (Same Answers)

9. Examine your answers for Exercises 6, 7, and 8.

   Fill in >, =, < to make these sentences true. Put the same symbol in both blanks of each sentence.

   a. When one factor in a product expression (>) 1, the product (>) the other factor.

   b. When one factor in a product expression (=) 1, the product (=) the other factor.

   c. When one factor in a product expression (<) 1, the product (<) the other factor.

(These answers can be written in any order)

A NEW PROPERTY OF RATIONAL NUMBERS: RECIPROCAL PROPERTY

Objective: To develop a new property of rational numbers, the Reciprocal Property, which was not possessed by the set of whole numbers.

Vocabulary: Reciprocal.

Suggested Teaching Procedure:

See notes on page 202.

Have the pupils do the exercises in the Exploration to discover a new property of the set of rational numbers, not possessed by the set of whole numbers.

A NEW PROPERTY OF RATIONAL NUMBERS: RECIPROCAL PROPERTY

You have seen that some properties of whole numbers are also properties of rational numbers. Do you think that rational numbers may have some properties which whole numbers do not have?

Exploration

1. Find r in each sentence.

a. $r = \frac{3}{4} \times \frac{4}{3}$ (1) c. $r = \frac{5}{2} \times \frac{2}{5}$ (1) e. $r = \frac{25}{3} \times \frac{3}{25}$ (1)

b. $r = \frac{7}{8} \times \frac{8}{7}$ (1) d. $r = \frac{9}{10} \times \frac{10}{9}$ (1) f. $r = \frac{100}{7} \times \frac{7}{100}$ (1)

2. What do you notice about the product in each part of Exercise 1? (It is 1)

3. Write the two factors in Exercise 1-a. What do you notice about them? (The numerator and the denominator are interchanged.)

4. Do you notice the same thing about the factors in Exercise 1-b through 1-f? (Yes)

Find the rational number n which makes each sentence true.

5. $\frac{2}{3} \times n = 1$ $(n = \frac{3}{2})$

6. $n \times \frac{8}{10} = 1$ $(n = \frac{10}{8})$

7. $1 = \frac{7}{5} \times n$ $(n = \frac{5}{7})$

8. $n \times \frac{12}{5} = 1$ $(n = \frac{5}{12})$

$\frac{4}{3}$ is called the reciprocal of $\frac{3}{4}$

$\frac{7}{8}$ is the reciprocal of $\frac{8}{7}$.

9. What is the product when a number is multiplied by its reciprocal? (1)

10. Find n in each sentence:

a. $n = 0 \times \frac{5}{8}$ (0) c. $n = \frac{7}{9} \times 0$ (0) e. $n = \frac{13}{18} \times \frac{0}{23}$ (0)

b. $n = \frac{3}{4} \times \frac{0}{2}$ (0) d. $n = \frac{0}{3} \times \frac{8}{5}$ (0) f. $n = 0 \times \frac{17}{10}$ (0)

11. What property of multiplication of rational numbers does Exercise 10 illustrate? (Any rational number multiplied by 0 is 0.)

12. If possible, find a rational number n which makes each sentence true:

a. $\frac{0}{3} \times n = 0$ c. $n \times 0 = 1$ (none possible)

b. $n \times \frac{0}{4} = 1$ d. $0 \times n = 0$

13. Is there a rational number which does not have a reciprocal? (Yes) Is there more than one such rational number? (No)

14. Can we state this property for rational numbers? For every rational number $\frac{a}{b}$, if a is not 0, and b is not 0, $\frac{a}{b} \times \frac{b}{a} = 1$. (Yes)

15. Could the property in Exercise 14 be stated this way? (Yes) Every rational number except 0 has a reciprocal.

16. Think of the set of whole numbers. Can you find a whole number n such that (No)

a. $5 \times n = 1$? b. $n \times 8 = 1$?

17. Does the set of whole numbers have the reciprocal property stated in Exercise 15? (No)

18. Find a rational number n such that the sentence in Exercise 16a is true. $\left(\frac{1}{3}\right)$ Do the same for the sentence in Exercise 16b. $\left(\frac{1}{8}\right)$

19. Do you see an easy way to find the reciprocal of a rational number? (Interchange the numerator and denominator.)

20. The measure of a rectangular region is 1. Find the measure of a side, if the measure of the other side is:

a. $\frac{3}{4}$ $\left(\frac{4}{3}\right)$ c. $\frac{9}{4}$ $\left(\frac{4}{9}\right)$ e. $\frac{11}{13}$ $\left(\frac{13}{11}\right)$

b. $\frac{7}{10}$ $\left(\frac{10}{7}\right)$ d. $1\frac{1}{2}$ $\left(\frac{2}{3}\right)$ f. 0.25 $\left(\frac{100}{25}\right)$

If the product of two rational numbers is 1,
each number is the reciprocal of the other.

Exercise Set 21

1. Mr. Brown bought 6.1 gallons of gas on Saturday and 7.9 gallons on Sunday. How many gallons of gas did Mr. Brown buy on the two days together? ($(6.1 + 7.9) = n$ Mr. Brown bought 14 gallons of gas.)

2. One jet averaged 659.49 miles per hour on its test flight. Another jet averaged 701.1 miles per hour. How much greater was the average speed of the second plane? ($701.1 - 659.49 = p$ The second jet traveled 41.61 miles per hour faster.)

3. If the average rainfall in a state is 2.7 inches per month, what will be the total rainfall for the year? ($12 \times 2.7 = r$ The total rainfall for the year is 32.4 inches.)

4. What is the area of a rectangular room whose sides are $12\frac{3}{4}$ feet and 15 feet long? ($12\frac{3}{4} \times 15 = s$ The area of the room is $191\frac{1}{4}$ sq. ft.)

5. Jim is $50\frac{1}{2}$ inches tall. Sally is $48\frac{3}{4}$ inches tall. How much taller is Jim than Sally? ($50\frac{1}{2} - 48\frac{3}{4} = p$ Jim is $1\frac{3}{4}$ inches taller.)

6. A recipe called for $1\frac{3}{4}$ cups oatmeal, and $1\frac{1}{2}$ cups flour, and $1\frac{1}{3}$ cups raisins. How many cups of dry ingredients were called for in the recipe? ($1\frac{3}{4} + 1\frac{1}{2} + 1\frac{1}{3} = t$ There were $4\frac{7}{12}$ cups of dry ingredients.)

7. Joel has planted $\frac{3}{4}$ of his garden in vegetables. $\frac{1}{2}$ of this section is planted in tomatoes. What part of the whole garden is planted in tomatoes? ($\frac{3}{4} \times \frac{1}{2} = x$ Joel has $\frac{3}{8}$ of his garden planted in tomatoes.)

8. If 1 day is $\frac{1}{7}$ of a week, what part of the week is 12 hours? ($\frac{1}{2} \times \frac{1}{7} = p$ Twelve hours are $\frac{1}{14}$ of a week.)

9. Allen drinks $1\frac{1}{2}$ cups of milk three times a day. How many cups of milk does he drink in one week? ($3 \times (1\frac{1}{2} \times 7) = n$ Allen drinks $31\frac{1}{2}$ cups of milk in a week.)

10. Eddie's house is $\frac{3}{8}$ mile from school. How far does he walk each day if he makes two round trips? How far does he walk each week? ($\frac{3}{8} \times 4 = a$. Eddie walks $1\frac{1}{2}$ miles each day $1\frac{1}{2} \times 5 = t$. Eddie walks $7\frac{1}{2}$ miles in one week.)

11. For the summer, Rick and Sam cut lawns for the neighbors. Together they charged 3 dollars an hour. They worked $7\frac{3}{4}$ hours each day. How much did they earn in one day? in one week (5 days)? ($3 \times 7\frac{3}{4} = n$ They earned $23\frac{1}{4}$ dollars per day. $23\frac{1}{4} \times 5 = a$. They earned $116\frac{1}{4}$ dollars per week.)

12. Bill is $2\frac{1}{3}$ years older than Bob. Bob is $3\frac{1}{2}$ years older than Jack. How much older is Bill than Jack? ($2\frac{1}{3} + 3\frac{1}{2} = n$ Bill is $5\frac{5}{6}$ years older or 5 years 10 months older than Jack.)

13. The measurements of the sides of a rectangular sheet of metal are 17.2 inches and 9.8 inches. What is the area of the sheet in square inches? ($17.2 \times 9.8 = a$ The area of the sheet is 168.56 sq. inches.)

14. California had 3.6 inches of rain in January, 5.1 inches in February, 5.8 inches in March, and 4.4 inches in April. The total amount for the year was 23.0 inches. How much rain fell during the other eight months?
($3.6+5.1+5.8+4.4=n$ $23-n=m$ 4.1 inches of rain fell in the other months.)

15. Mrs. Jackson baked $2\frac{1}{2}$ dozen cookies. For lunch Helen ate $\frac{1}{4}$ dozen, Janet ate $\frac{1}{3}$ dozen, Dotty ate $\frac{1}{6}$ dozen, and Ellen ate $\frac{1}{12}$ dozen. How many dozen cookies were left. ($\frac{1}{4}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+n=2\frac{1}{2}$ There were $1\frac{2}{3}$ dozen left.)

COMPUTING PRODUCTS OF RATIONAL NUMBERS USING DECIMALS

Objective: To extend the decimal system of notation to numbers including hundred-thousandths.

Materials: Pocket chart and cards on page 88

Vocabulary: Expanded notation.

Suggested Teaching Procedure:

> In Chapter 1, the decimal notation for numbers has been developed to thousandths. In this section, the system is extended to hundred-thousandths. Since the pupils now can multiply rational numbers, review the decimal system of notation and let them discover what the name of the fourth and fifth places to the right of the decimal point should be. Some pupils may wish to go farther and name the sixth and seventh places also.
>
> Write a numeral such as 238.467 on the board.

How do you read this numeral? What number does the 2 represent? the 3? etc. What is the place-value name of the place in which 2 is written? in which 3 is written? etc.

> Write the place-value name above each digit.

How does each place value compare with the place value to its left? ($10 = \frac{1}{10} \times 100$, $1 = \frac{1}{10} \times 10$, $0.1 = \frac{1}{10} \times 1$, etc.)

Suppose you have this numeral: 238.4671. How can you find what number the 1 represents? ($\frac{1}{10} \times \frac{1}{1000}$) What should be its place-value? (It should be $\frac{1}{10} \times \frac{1}{1000}$. It should be "ten-thousandths.)

Could you continue to find the names for new places in the same way? What should be the place-value for a digit written after the 1 in 238.4671? ($\frac{1}{10} \times \frac{1}{10,000} = \frac{1}{100,000}$). What is another numeral for 100? for 1000? for 10,000? for 100,000?

> If pupils cannot answer, suggest using exponents.

Objective: To find products of rational numbers named by decimals by using (a) computational procedure for multiplying whole numbers and (b) the law of exponents for multiplication.

Material: Pocket charts and cards page 88

Suggested Teaching Procedure:

To this point, pupils have multiplied rational numbers named by decimals by thinking about the numerator and denominator of the fraction form for the decimal. In this section use is made of the properties of rational numbers to show that such products can be found by (a) using the computational procedure for multiplying whole numbers, and (b) using the law of exponents for multiplying powers of the same base to help determine the correct place value of the product.

Some pupils will probably discover the relation which is often stated as a "rule" for placing the decimal point in a product. However, it is important that the suggested vertical form for multiplication be followed so that they see that the relation they have observed is an outcome of the system of decimal notation rather than a mechanical "rule".

Before teaching this section you may find it advisable to review the laws of exponents studied in Chapter 1.

COMPUTING PRODUCTS OF RATIONAL NUMBERS USING DECIMALS

Exploration
(Extending Decimal Notation)

1. You know that 45.687 means

   $(4 \times 10) + (5 \times 1) + (6 \times \frac{1}{10}) + (8 \times \frac{1}{100}) + (7 \times \frac{1}{1000})$

   Suppose you see the numeral 45.6872. What number does the digit 2 represent? $(2 \times \frac{1}{10^4})$ What should be the place-value name? (Ten-thousandths) In the decimal system of numeration, the value of each place is $\frac{1}{10}$ the value of the place on its left. For example,

   $$0.1 = \frac{1}{10}$$
   $$0.01 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$
   $$0.001 = \frac{1}{10} \times \frac{1}{100} = \frac{1}{1,000}.$$

   So the value of the next place is $\frac{1}{10} \times \frac{1}{1,000}$, or $\frac{1}{10,000}$. The digit 2 in the numeral above represents $2 \times \frac{1}{10,000}$, or 2 ten-thousandths.

2. Suppose the numeral is 45.68723. What number does the 3 represent? $(3 \times \frac{1}{10^5})$ What should be the place-value name? (Hundred-thousandths)

3. Does the chart below agree with your answer? (Yes)

Millions	Hundred-thousands	Ten-thousands	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
					4	5.	6	8	7	2	3

Exercise Set 22

1. Express the meaning of the numeral 97.04682 in expanded notation. $\left[(9 \times 10^1) + (7 \times 1) + (0 \times \frac{1}{10^1}) + (4 \times \frac{1}{10^2}) + (6 \times \frac{1}{10^3}) + (8 \times \frac{1}{10^4}) + (2 \times \frac{1}{10^5})\right]$

2. Jack wrote his answer this way:

$$(9 \times 10) + (7 \times 1) + (0 \times \frac{1}{10}) + (4 \times \frac{1}{100}) + (6 \times \frac{1}{1000}) +$$
$$(8 \times \frac{1}{10,000}) + (2 \times \frac{1}{100,000})$$

Bill wrote this:

$$(9 \times 10) + (7 \times 1) + (0 \times \frac{1}{10}) + (4 \times \frac{1}{10^2}) + (6 \times \frac{1}{10^3}) +$$
$$(8 \times \frac{1}{10^4}) + (2 \times \frac{1}{10^5})$$

Whose answer was correct? (both)

3. Show that $0.2761 = \frac{2761}{10,000}$ by first writing 0.2761 in expanded notation. $\left[(0 \times 10^1) + (2 \times \frac{1}{10^1}) + (7 \times \frac{1}{10^2}) + (6 \times \frac{1}{10^3}) + (1 \times \frac{1}{10^4})\right]$

4. Write the decimal name for

   a. 234 thousandths (0.234)

   b. 17 ten-thousandths (0.0017)

   c. 346 hundred-thousandths (0.00346)

5. Write the fraction form for

   a. 0.2567 $\left(\frac{2567}{10,000}\right)$

   b. 0.01682 $\left(\frac{1682}{100,000}\right)$

   c. 32.5678 $\left(\frac{325678}{10,000}\right)$

6. Suppose you wanted to separate a unit square region into square regions to show ten-thousandths. How many congruent segments should you make on each side of the square? (100)

Exploration

You know that you can multiply rational numbers with decimal names by thinking about their fraction names. Is there another way?

Consider the product 4.53 × 0.007

a. $4.53 = \frac{453}{100} = 453 \times \frac{1}{100} = 453 \times \frac{1}{10^2}$

b. $0.007 = \frac{7}{1000} = 7 \times \frac{1}{1000} = 7 \times \frac{1}{10^3}$

c. So, $4.53 \times 0.007 = (453 \times \frac{1}{10^2}) \times (7 \times \frac{1}{10^3})$

d. $= 453 \times (\frac{1}{10^2} \times 7) \times \frac{1}{10^3}$

e. $= 453 \times (7 \times \frac{1}{10^2}) \times \frac{1}{10^3}$

f. $= (453 \times 7) \times (\frac{1}{10^2} \times \frac{1}{10^3})$

g. $= (453 \times 7) \times \frac{1 \times 1}{10^2 \times 10^3}$

h. $= 3171 \times \frac{1}{10^5}$

i. $= 3171 \times \frac{1}{100,000}$

j. $= \frac{3171}{100,000} = 0.03171$

1. Explain line a and line b. ($\frac{453}{100}$ is fraction form for 4.53. $\frac{7}{1000}$ is fraction form for 0.007)
2. What property of rational numbers is used in line d ? in line e ? in line f ? (d. associative e. commutative f. associative)
3. What is done in line g ? ($\frac{1}{10^2} \times \frac{1}{10^3}$ has been renamed)
4. What law of exponents is used in line h ? (To multiply two powers of the same base, add the exponents)
5. How is the 100,000 obtained in line i ? (10 × 10 × 10 × 10 × 10 = 100,000)

Lines f and g show that you can find the product of two rational numbers named by decimals by a) multiplying as though they were whole numbers and b) placing a decimal point to indicate the correct place value.

The first step is familiar. How can you tell where the decimal point should be? Look at line k below.

k. $4.53 \times 0.007 = (453 \times \frac{1}{10^2}) \times (7 \times \frac{1}{10^3})$

$= (453 \times 7) \times (\frac{1}{10^5})$

$= 0.03171.$

Do you see an easy way to tell what each exponent should be? Is there an easy way to decide how many digits in the product there should be to the right of the decimal point?

6. Write these products as shown in line k.

a. 2.46×3.1
$(246 \times \frac{1}{10^2}) \times (31 \times \frac{1}{10^1}) = (246 \times 31) \times (\frac{1}{10^3})$

b. 0.513×9.2
$(513 \times \frac{1}{10^3}) \times (92 \times \frac{1}{10^1}) = (513 \times 92) \times (\frac{1}{10^4})$

c. 1.68×0.005
$(168 \times \frac{1}{10^2}) \times (5 \times \frac{1}{10^3}) = (168 \times 5) \times (\frac{1}{10^5})$

d. 6.2×1.049
$(62 \times \frac{1}{10^1}) \times (1049 \times \frac{1}{10^3}) = (62 \times 1049) \times (\frac{1}{10^4})$

Since you find the product by first multiplying as with whole numbers, it is convenient to arrange your work in vertical form and record your thinking like this:

$$\begin{array}{rl}
1.049 & 1049 \times \frac{1}{10^3} \\
\underline{6.2} & 62 \times \frac{1}{10} \\
2\,098 & \\
\underline{62\,94} & \\
6.5038 & 65038 \times \frac{1}{10^4}
\end{array}$$

Exercise Set 23

Use the vertical form as shown to find r.

1. r = 3.25 × 0.04 (0.1300)
2. r = 6.17 × 0.29 (1.7893)
3. r = 0.048 × 1.46 (0.07008)
4. r = 3.1 × 0.307 (0.9517)
5. r = 58 × 7.23 (419.34)
6. r = 0.96 × 7.7 (7.392)
7. r = 0.18 × 0.056 (0.01008)
8. r = 3.68 × 1.42 (5.2256)
9. r = 19.03 × 8.5 (161.755)

RATIONAL NUMBERS WITH DECIMAL NAMES

Objectives: To direct attention to the fact that some, but not all, rational numbers can be named by decimals.

To note that this fact can be explained by considering the prime factorization of 10 and powers of 10.

Suggested Teaching Procedure:

Every rational number can be named by some fraction whose numerator and denominator are whole numbers--in fact, by many such fractions. At this stage we have decimal names for some, but not all, rational numbers. For example, $\frac{1}{4} = \frac{25}{100}$, and therefore 0.25 is a decimal name for $\frac{1}{4}$. So are 0.250 and 0.2500.

But consider the rational number whose simplest name is $\frac{1}{3}$. If it is to have a decimal name, then the decimal will have a fraction form with denominator 10, or 100, or 1000, or some higher power of 10. But is there any whole number n for which the sentence $\frac{1}{3} = \frac{n}{10}$ true? or the sentence $\frac{1}{3} = \frac{n}{100}$? or $\frac{1}{3} = \frac{1}{1000}$? You know that there is no such whole number n. The reason can be found by considering two familiar facts:

(a) Since $\frac{1}{3}$ is the simplest name for the number, other fraction names for the number can be found only by multiplying the numerator and denominator of $\frac{1}{3}$ by some counting number.

(b) The prime factorizations for powers of 10 go like this:

$$10 = 2 \times 5$$
$$100 = 2 \times 2 \times 5 \times 5$$
$$1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$$
$$10000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5$$

We can note the relation between the factorization of the denominator of a given fraction and the place value of its simplest decimal name. In $\frac{1}{4}$, since $4 = 2 \times 2$, the power of 10 in the denominator of the fraction form of its decimal name must have a factorization with two factors of 2, which requires that it be 10^2. In $\frac{1}{16}$, $16 = 2 \times 2 \times 2 \times 2$, so the decimal must have a place value of $\frac{1}{10^4}$. In $\frac{1}{12}$, $12 = 2 \times 2 \times 3$, and no power of 10 has the factor 3, so $\frac{1}{12}$ has no decimal name.

You may see the expression

$$\frac{1}{3} = .33\frac{1}{3}$$

The numeral $.33\frac{1}{3}$ is not a decimal numeral, however. It is a mixed form which is an abbreviation for $.33 + (\frac{1}{3} \times \frac{1}{100})$.

RATIONAL NUMBERS WITH DECIMAL NAMES

You know that the decimal name for a number can be written easily if the fraction name has denominator 10, 10^2, 10^3, etc. What about numbers whose fraction names have other denominators? Can you name the number $\frac{1}{4}$ by a fraction with denominator 10, or 100, or 1000? Since $\frac{1}{4} = \frac{25}{100}$, $\frac{1}{4}$ and 0.25 name the same number.

Exercise Set 24

1. If possible, for these numbers find fraction names with whole number numerators and with denominator 10, 100, 1000, or 10,000.

   a. $\frac{3}{4}$ $\left(\frac{75}{100}\right)$
   b. $\frac{5}{8}$ $\left(\frac{625}{1000}\right)$
   c. $\frac{2}{3}$ (not possible)
   d. $\frac{3}{5}$ $\left(\frac{6}{10}\right)$
   e. $\frac{1}{16}$ $\left(\frac{625}{10,000}\right)$
   f. $\frac{2}{5}$ $\left(\frac{4}{10}\right)$
   g. $\frac{1}{4}$ $\left(\frac{25}{100}\right)$
   h. $\frac{5}{6}$ (not possible)
   i. $\frac{3}{8}$ $\left(\frac{375}{1000}\right)$
   j. $\frac{1}{7}$ (not possible)

2. You should have found fraction names with whole number numerators for all but three of the numbers in Exercise 1. Explain why you could not find fraction names with whole number numerators for these three numbers. (Hint: Find the prime factorization of 10, 100, 1000, and 10,000. Find the prime factorization of the denominators of the fractions for the three numbers.) (c. 3 is not a factor of 10, 100, 1000, or 10,000)
(h. $6 = 2 \times 3$, $10 = 2 \times 5$, $100 = 2^2 \times 5^2$, so no power of 10 has 3 as a factor)
(j. 7 is not a factor of 10, so it is not a factor of 100, 1000 or 10,000)

3. Write decimal names for the other seven numbers.
(a: 0.75, b: 0.625 d: 0.6, e: 0.0625, f: 0.4, g: 0.25, i: 0.375)

4. In your answers for Exercise 1, which fractions have the same denominator? (a and g, d and f, b and i)

USING THE DISTRIBUTIVE PROPERTY

Objective: To suggest another way to find the product of some rational numbers by applying the distributive property.

Materials: Rectangular regions.

Suggested Teaching Procedure:

For computing the product of two rational numbers, both greater than 1 and one of the numbers a whole number, an easier procedure for computation is usually found by applying the distributive property than by finding fraction names for both numbers. This page is included to suggest this possibility to the pupils.

Use the upper half of page 231 of pupil material for class discussion. Children may then do exercises 1, 2, and 3 on pupil page 232 independently.

USING THE DISTRIBUTIVE PROPERTY

You have found products of rational numbers by using their fraction names.

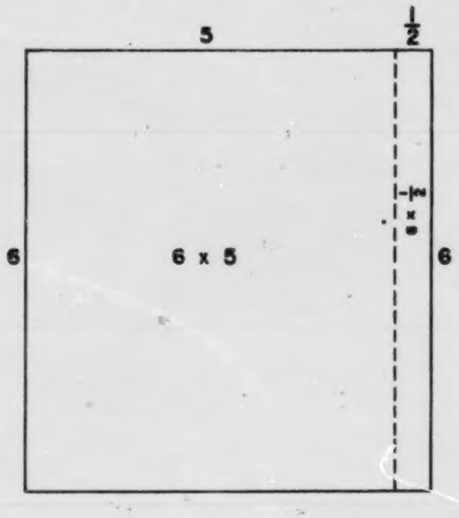
First way $6 \times 5\frac{1}{2} = \frac{6}{1} \times \frac{11}{2} = \frac{66}{2} = 33$

Since $5\frac{1}{2} = 5 + \frac{1}{2}$, you could also use the distributive property.

Second way

$$\begin{aligned} 6 \times 5\tfrac{1}{2} &= 6 \times (5 + \tfrac{1}{2}) \\ &= (6 \times 5) + (6 \times \tfrac{1}{2}) \\ &= 30 + \tfrac{6}{2} \\ &= 30 + 3 \\ 6 \times 5\tfrac{1}{2} &= 33 \end{aligned}$$

You can illustrate the second way by using rectangular regions.



Exercise Set 25

1. Find n by using the distributive property. Write each product in simplest form.

a. $n = 8 \times 2\frac{3}{4}$ (22)

b. $n = 5\frac{1}{3} \times 7$ $(37\frac{1}{3})$

c. $n = 4 \times 16\frac{1}{2}$ (66)

d. $n = 12\frac{1}{8} \times 24$ (291)

e. $n = 6\frac{3}{4} \times 30$ $(202\frac{1}{2})$

f. $n = 14 \times 9\frac{3}{7}$ (132)

2. Draw a rectangular region and separate it to illustrate one of the products in Exercise 1. (Answers will vary)

3. Find the products in Exercise 1 by using $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. Write the simplest name for each product. Which way is easier? (a: 22, b: $37\frac{1}{3}$, c: 66, d: 291, e: $202\frac{1}{2}$, f: 132)

ESTIMATING PRODUCTS

Objective: To encourage children to look critically at the size of the rational numbers on which they are operating and estimate the result.

Material: Number line.

Vocabulary: Estimate.

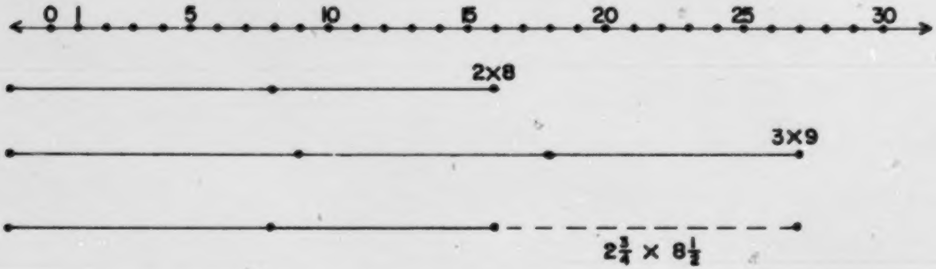
Suggested Teaching Procedure:

When pupils are computing with rational numbers they frequently fail to detect gross errors in their own work because they do not think critically about the size of the numbers involved. The purpose of this discussion and Exercise Set is to direct their attention to the facts that

(a) any rational number lies between two consecutive whole numbers, and

(b) the product (or sum) of two rational numbers lies between the product (or sum) of the two lower whole numbers and the product (or sum) of the two higher whole numbers.

You may want to use a number line to show the interval in which $2\frac{3}{4} \times 8\frac{1}{2}$ must lie.



ESTIMATING PRODUCTS

When you are multiplying rational numbers, it is a good idea to estimate the product first.

Consider the product

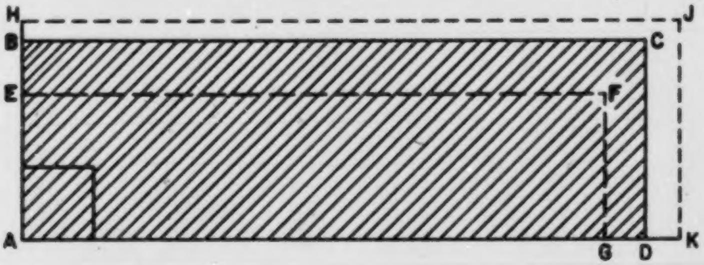
$2\frac{3}{4} \times 8\frac{1}{2}$. How large should the product be?

$2\frac{3}{4} > 2$ $\qquad$ $8\frac{1}{2} > 8$

so $(2\frac{3}{4} \times 8\frac{1}{2}) > (2 \times 8)$. The product must be greater than 16.

$2\frac{3}{4} < 3$ $\qquad$ $8\frac{1}{2} < 9$

So $2\frac{3}{4} \times 8\frac{1}{2}$ is between 16 and 27.



In the drawing, the unit square region and the $2\frac{3}{4}$ by $8\frac{1}{2}$ region are in dark lines.

Name the 2 by 8 region. (AEFG)

Name the 3 by 9 region. (AHJK)

Exercise Set 26

Which of the answers below *may* be right? Which ones *must* be wrong? Answer by finding two whole numbers between which the product must be.

1. $3\frac{1}{2} \times 5\frac{7}{8} = 20\frac{9}{16}$

2. $12\frac{7}{10} \times 3\frac{5}{8} = 32\frac{41}{80}$

3. $5\frac{3}{4} + 24\frac{7}{8} = 32\frac{5}{8}$

4. $2\frac{3}{4} \times 5280 = 15{,}840$

5. $\frac{7}{8} \times 6\frac{1}{2} = 6\frac{3}{4}$

6. $42\frac{7}{10} + 26\frac{8}{15} = 69\frac{1}{10}$

(1, 5, 6 *may* be right; 2, 3, 4 *must* be wrong.)

Between what two whole numbers must each sum or product be?

7. $2\frac{3}{4} + 3\frac{1}{2}$ (5 - 7)

8. $7\frac{9}{10} + 15\frac{3}{7}$ (22 - 24)

9. $10\frac{4}{9} + 12\frac{7}{15}$ (22 - 24)

10. $128\frac{3}{17} + 245\frac{125}{147}$ (373 - 375)

11. $56.7 + 38.54$ (94 - 96)

12. $8\frac{3}{4} \times 3\frac{2}{7}$ (24 - 36)

13. $9\frac{4}{13} \times 11\frac{5}{19}$ (99 - 120)

14. $6\frac{5}{23} \times 8\frac{46}{47}$ (48 - 63)

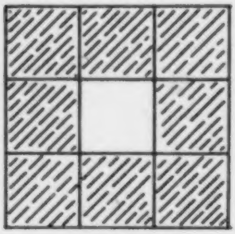
15. $15\frac{1}{2} \times 10\frac{51}{53}$ (150 - 176)

16. 7.28×0.34 (0 - 8)

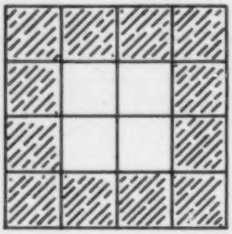
17. Which can you estimate more closely by the method described: the sum of two numbers or the product of the same two numbers? (sum)

Exercise Set 27

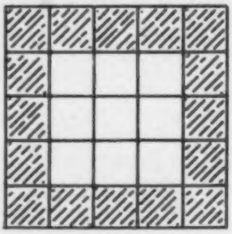
1. Below are three unit squares. Each is separated into smaller congruent squares, and the border squares are shaded.



A



B



C

   a. What is the measure of a side of the unshaded square region of each? (A: $\frac{1}{3}$, B: $\frac{1}{2}$ or $\frac{2}{4}$, C: $\frac{3}{5}$)

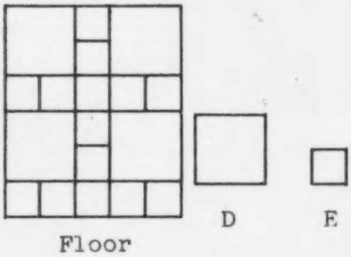
   b. What is the measure of the unshaded square region of each? (A: $\frac{1}{9}$, B: $\frac{1}{4}$, C: $\frac{9}{25}$)

   c. What is the measure of the shaded region of each? (A: $\frac{8}{9}$, B: $\frac{3}{4}$, C: $\frac{16}{25}$)

2. The sides of the unit squares in Exercise 1 are separated into 3 congruent parts, 4 congruent parts, and 5 congruent parts. Draw another unit square; separate its sides into 6 congruent parts. Separate the unit square region into smaller square regions, and shade the border region.

   Answer questions a, b, and c, from Exercise 1 about your unit square. (a. $\frac{4}{6}$ or $\frac{2}{3}$ b. $\frac{16}{36}$ or $\frac{4}{9}$ c. $\frac{20}{36}$ or $\frac{5}{9}$)

3. A tile floor, pictured to the right, is made of tile of the two sizes shown. The measure of tile D is 1 and tile E is $\frac{1}{4}$.

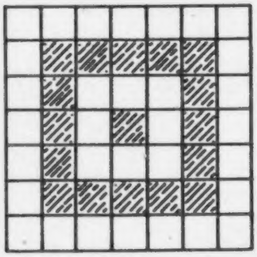


a. What is the measure of the part of the floor covered with small tile? $\left(\frac{14}{4}\text{ or }3\frac{1}{2}\right)$

b. What is the measure of the part of the floor covered with the large tile? (4)

c. What is the measure of the floor? $\left(7\frac{1}{2}\right)$

4. The square pattern to the right was made by fitting together black and white square tile. The pattern has been used by artists for many years.



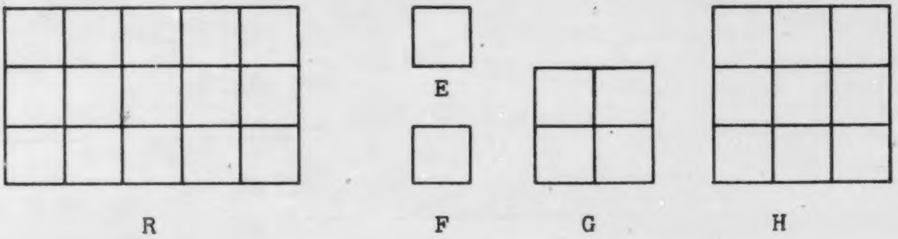
a. If the measure of the whole region is 1, what is the measure of each tile? $\left(\frac{1}{49}\right)$

b. What is the measure of the white outer border? $\left(\frac{24}{49}\right)$

c. What is the measure of the white inner border? $\left(\frac{8}{49}\right)$

d. What is the measure of the black outer border? $\left(\frac{16}{49}\right)$

5. Below are two arrangements of 15 squares. On the left, they are arranged in a rectangle; on the right, they are arranged in four squares.



a. If the measure of the rectangular region is 1, what is the measure of each square region? $\left(E: \frac{1}{15}, F: \frac{1}{15}, G: \frac{4}{15}, H: \frac{9}{15}\right)$

b. If the measure of the square region H is 1, what is the measure of the rectangular region? $\left(\frac{15}{9} \text{ or } 1\frac{2}{3}\right)$

c. If the measure of square region G is 1, what is the measure of the rectangular region? $\left(\frac{15}{4} \text{ or } 3\frac{3}{4}\right)$

6. Write each of the following in its simplest form:

a. $\frac{4}{3}$ $\left(1\frac{1}{3}\right)$　d. $\frac{64}{9}$ $\left(7\frac{1}{9}\right)$　g. $\frac{25}{15}$ $\left(1\frac{2}{3}\right)$

b. $7\frac{6}{5}$ $\left(8\frac{1}{5}\right)$　e. $12\frac{8}{4}$ (14)　h. $\frac{399}{1000}$ $\left(\frac{399}{1000}\right)$

c. $17\frac{3}{6}$ $\left(17\frac{1}{2}\right)$　f. $\frac{37}{100}$ $\left(\frac{37}{100}\right)$　i. $\frac{16}{16}$ (1)

Exercise Set 28

1. Supply the missing numerators:

a. $1\frac{2}{3} = \frac{(5)}{3}$

b. $5\frac{1}{8} = \frac{4(9)}{8}$

c. $1 = \frac{(5)}{5}$

d. $\frac{16}{100} = \frac{(160)}{1000}$

e. $17\frac{4}{8} = \frac{(140)}{8}$

f. $\frac{2}{3} = \frac{(8)}{12}$

2. Complete the chart below:

Rectangle	Measurement of Adjacent Sides	Perimeter	Area of Regions
A	$4\frac{1}{3}$ ft. by $6\frac{1}{2}$ ft.	($21\frac{2}{3}$ ft.)	($28\frac{1}{6}$ sq. ft)
B	12.75 ft. by 18.18 ft.	(63.10 ft.)	(239.7 sq. ft)
C	9 ft. by 3 ft.	(24 ft)	(27 sq. ft)

a. How much greater is the measure of rectangular region A than C ? ($1\frac{1}{6}$)

b. How much greater is the measure of rectangular region B than C ? (212.7)

c. How much less is the measure of the perimeter of A than C ? ($2\frac{1}{3}$)

d. How much less is the measure of the perimeter of C than B ? (39.10)

3. Arrange from least to greatest:

$\frac{1}{7}$; $\frac{2}{2}$; $\frac{2}{3}$; 0.5; $\frac{3}{4}$; $1\frac{1}{5}$ ($\frac{1}{7}$; 0.5; $\frac{2}{3}$; $\frac{3}{4}$; $\frac{2}{2}$; $1\frac{1}{5}$)

4. **Arrange from least to greatest:**

0.5; 0.49; 1.8; 7.09; 0.001; $\frac{4}{10}$

(0.001; $\frac{4}{10}$; 0.49; 0.5; 1.8; 7.09)

5. **Subtract:**

a. $\begin{array}{r} 5\frac{4}{5} \\ 1\frac{2}{5} \\ \hline (4\frac{2}{5}) \end{array}$ c. $\begin{array}{r} 29\frac{1}{8} \\ 17\frac{3}{8} \\ \hline (11\frac{3}{4}) \end{array}$ e. $\begin{array}{r} 90\frac{1}{3} \\ 46\frac{1}{2} \\ \hline (43\frac{5}{6}) \end{array}$

b. $\begin{array}{r} 14 \\ 5\frac{1}{6} \\ \hline (8\frac{5}{6}) \end{array}$ d. $\begin{array}{r} 57\frac{9}{10} \\ 39\frac{4}{5} \\ \hline (18\frac{1}{10}) \end{array}$ f. $\begin{array}{r} 74\frac{3}{8} \\ 65\frac{5}{6} \\ \hline (8\frac{13}{24}) \end{array}$

6. **Find the simplest fraction name for:**

a. $\frac{5}{2} \times \frac{1}{4}$ $\left(\frac{5}{8}\right)$ d. $1\frac{1}{2} \times 2\frac{2}{3}$ (4)

b. $7 \times 5\frac{2}{3}$ $\left(39\frac{2}{3}\right)$ e. $\frac{5}{8} \times 10$ $\left(6\frac{1}{4}\right)$

c. $\frac{4}{5} \times \frac{4}{5}$ $\left(\frac{16}{25}\right)$ f. $4\frac{5}{6} \times 3\frac{1}{2}$ $\left(16\frac{11}{12}\right)$

7. **Add:**

a. $\begin{array}{r} 1\frac{1}{5} \\ 2\frac{3}{5} \\ \hline (3\frac{4}{5}) \end{array}$ c. $\begin{array}{r} 9\frac{5}{6} \\ 8\frac{5}{6} \\ \hline (18\frac{2}{3}) \end{array}$ e. $\begin{array}{r} 101\frac{1}{2} \\ 47\frac{2}{5} \\ \hline (148\frac{9}{10}) \end{array}$

b. $\begin{array}{r} 7\frac{3}{8} \\ 6\frac{5}{8} \\ \hline (14) \end{array}$ d. $\begin{array}{r} 34\frac{2}{3} \\ 15\frac{3}{4} \\ \hline (50\frac{5}{12}) \end{array}$ f. $\begin{array}{r} 69\frac{5}{6} \\ 82\frac{7}{8} \\ \hline (152\frac{17}{24}) \end{array}$

Exercise Set 29

1. **Name the following numbers by fractions:**

a. 0.7 $\left(\frac{7}{10}\right)$ c. 2.57 $\left(\frac{257}{100}\right)$ e. 0.072 $\left(\frac{72}{1000}\right)$

b. 0.04 $\left(\frac{4}{100}\right)$ d. 3.6 $\left(\frac{36}{10}\right)$ f. 1.25 $\left(\frac{125}{100}\right)$

2. **Name the following numbers by decimals:**

a. $\frac{9}{10}$ (0.9) c. $\frac{125}{100}$ (1.25) e. $\frac{43}{10}$ (4.3)

b. $\frac{49}{100}$ (0.49) d. $\frac{10}{10}$ (1.0) f. $\frac{417}{1000}$ (0.417)

3. **Find n.**

a. $7.29 + 0.7 = n$ ($n = 7.99$) c. $0.37 + 0.8973 = n$ ($n = 1.2673$)

b. $31 + 2.59 = n$ ($n = 33.59$) d. $5.235 + 4 + 6.25 = n$ ($n = 15.485$)

4. **Find n.**

a. $0.90 - 0.4 = n$ ($n = 0.50$) c. $4.205 - 1.7416 = n$ ($n = 2.4634$)

b. $6.7 - 4.25 = n$ ($n = 2.45$) d. $47 - 0.478 = n$ ($n = 46.522$)

5. a. $0.3 \times 0.4 = n$ ($n = 0.12$) d. $48.8 \times 0.56 = n$ ($n = 27.328$)

b. $7.03 \times 0.9 = n$ ($n = 6.327$) e. $0.94 \times 6.8 = n$ ($n = 6.392$)

c. $0.78 \times 0.5 = n$ ($n = 0.390$) f. $3.42 \times 8.6 = n$ ($n = 29.412$)

Exercise Set 30

1. When the Smith family left on their vacation, the odometer read 19,628.6 miles. When they returned, it read 22,405.3. How many miles had the Smiths traveled? (19628.6 + n = 22405.3
The Smiths traveled 2776.7)

2. Mrs. Williams bought four pieces of steak weighing 4.7 pounds, 5.2 pounds, 5.3 pounds, and 3.8 pounds. At 99¢ per pound, how much will the four pieces cost?
(4.7 + 5.2 + 5.3 + 3.8) × 0.99 = n The meat will cost 18.81 dollars)

3. Jack's mother bought his fall clothes on sale. Shoes originally priced $7.89 were marked $\frac{1}{3}$ off. A suit originally priced $15.96 was marked $\frac{1}{4}$ off. A coat originally priced $19.98 was marked $\frac{1}{2}$ off. How much money did Jack's mother save? $(\frac{1}{3} \times 7.89) + (\frac{1}{4} \times 15.96) + (\frac{1}{2} \times 19.98) = n$
Jack's mot[illegible] saved 16.61 dollars)

4. When Mark pulled his lobster traps, he had 9 lobsters each weighing $1\frac{1}{4}$ pounds, 13 lobsters each weighing $1\frac{1}{2}$ pounds, and 8 lobsters each weighing $1\frac{3}{4}$ pounds. How many pounds of lobster did he pull? $((9 \times 1\frac{1}{4}) + (13 \times \frac{1}{2}) + (8 \times 1\frac{3}{4}) = n$
Mark pulled $44\frac{3}{4}$ pounds of lobster.)

5. The Ward's house is 42.8 feet by 68.5 feet. Their land is 105.5 feet by 236.2 feet. How many square feet of land do they have surrounding their house?
((105.5 × 236.2) - (42.8 × 68.5) = n. They have 21987.30 feet surrounding their house.)

6. One day, Helen and Rosemary were each given a guinea pig. Helen's guinea pig weighed 0.60 pounds and gained 0.07 pounds each day. Rosemary's guinea pig weighed 0.48 pounds, but ate more, and gained 0.09 pounds each day. Whose guinea pig was the heavier a week later? How much heavier? [(0.60 + (7 × 0.07) = n; n = 1.09) (0.48 + (7 × 0.09) = p; p = 1.11)
(1.11 - 1.09 = r; r = 0.02) (Rosemary's pig was 0.02 pounds heavier)]

7. In a swimming test, Dan stayed under water 2.3 times as long as Charlie. Charlie stayed under water 19.8 seconds. How long did Dan stay under water? (2.3 x 19.8 = n. Don stayed under water 45.54 seconds)

8. Races are sometimes measured in meters. A meter is 1.094 yards. What is the difference in yards between a 50 meter race and a 100 meter race? [(100 x 1.094) - (50 x 1.094) = p The difference is 54.7 yards.]

9. Paul weighs 40 pounds. Jerry weighs $1\frac{7}{8}$ times as much as Paul. Mike weighs $1\frac{3}{5}$ times as much as Jerry. How much do Jerry and Mike each weigh? ($1\frac{7}{8} \times 40 = n$. Jerry weighs 75 pounds. $1\frac{3}{5} \times 75 = p$. Mike weighs 120 pounds)

10. Ethel likes to collect colored rocks for her rock garden, but she can carry only 18 pounds of rock in her basket. If she puts in more, the basket will break. She puts six colored rocks in her basket. The first weighs 3.4 pounds, the next three weigh 3.1 pounds apiece, and the last two rocks weigh 2.6 pounds apiece. Will she break her basket? Explain. (3.4 + (3 x 3.1) + (2 x 2.6) = n. No, she will not breek her basket. The rocks weigh only 17.9 pounds.)

Chapter 3

SIDE AND ANGLE RELATIONSHIPS OF TRIANGLES

PURPOSE OF UNIT

1. To provide the pupil with the opportunity to become aware of different kinds of triangles.

2. To provide the pupil with the opportunity to compare sides and angles of triangles first by tracing, then by using straightedge and compass.

3. To provide the pupil with the opportunity to explore the properties of different kinds of triangles.

4. To develop the following understandings and skills:
    a. An isosceles triangle has at least two angles which are congruent to each other. These two congruent angles are opposite the sides which are congruent to each other.
    b. Equilateral triangles have three angles congruent to each other. Equilateral triangles are a subset of isosceles triangles; since they have three congruent angles they necessarily have two congruent angles.
    c. In scalene triangles the longest side lies opposite the largest angle and the shortest side lies opposite the smallest angle.

Materials Needed:

Teacher: Chalkboard and chalk, strips and fasteners, chalkboard compass or string compass, scissors

Pupil: Tracing paper, paper and pencil, strips and fasteners, compass and straightedge, scissors

Vocabulary: Relations (in the title of this section), and scalene

In these activities, children will be comparing sides, angles, and triangles by using tracings. The construction work will need to be done carefully so that the models will be as nearly congruent as is possible.

You may want to introduce this work by recalling with the children that they have just learned about many things in geometry such as congruent segments, angles and triangles, how to compare the size of angles, and how to use the compass and straightedge.

The children will now be working with special kinds of triangles. The isosceles and equilateral triangles have been previously introduced.

OVERVIEW

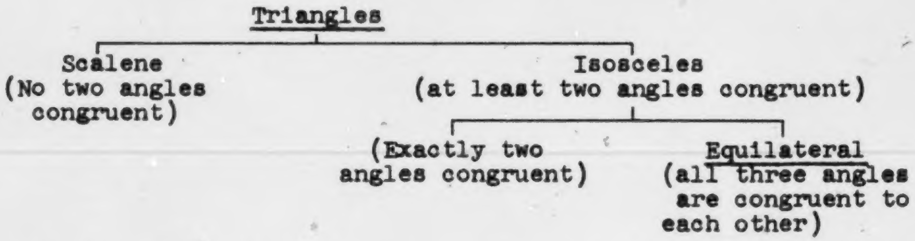
This unit, like the two geometry units preceding it in Grade V, Congruence of Common Geometric Figures and Measurement of Angles, is designed to provide the pupil with the opportunity to study different kinds of triangles and the interrelations of their sides and angles. The pupil first makes comparisons by the use of tracings. He then proceeds to make these comparisons by carefully constructing geometric figures using straightedge and compass. Since he has learned about congruent segments, congruent angles and congruent triangles and how to compare the sizes of angles, he is ready to explore properties of different kinds of triangles. He learns the following facts:

1. In an isosceles triangle at least two angles are congruent to each other.

2. In an equilateral triangle all three angles are congruent to each other.

3. In a scalene triangle, no two angles are congruent to each other.

The facts listed as 1, 2, and 3 may be summarized in the following chart.


Triangles
Scalene
(No two angles congruent)
Isosceles
(at least two angles congruent)
(Exactly two angles congruent)
Equilateral
(all three angles are congruent to each other)

4. The largest angle of a scalene triangle lies opposite the longest side and the smallest angle lies opposite the shortest side.

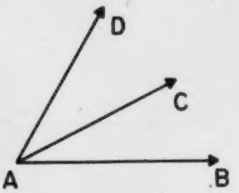
5. This chart may be helpful. If triangles are classified as to relative lengths of sides this classification shows the relationship.

Triangles

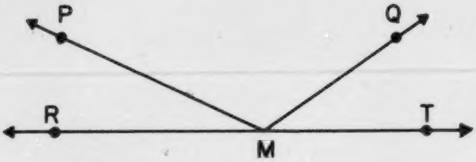
- Scalene (No two sides congruent)
- Isosceles (At least two congruent sides)
  - (Exactly two congruent sides)
  - Equilateral (all three sides are congruent to each other)

On basis of lengths of sides, triangles are either scalene or isosceles. If they are isosceles they may be equilateral also.

The pupil next explores the sum of the measures of two angles which have a common vertex, a common ray and interiors which do not intersect.



He proceeds rapidly to extend the situation to more than two angles and further, to the case where one ray of the first angle and one ray of the last angle lie on a line.



$\overrightarrow{MR}$ and $\overrightarrow{MT}$ lie on $\overleftrightarrow{RT}$ and $m\angle TMQ + m\angle QMP + m\angle PMR = 180$, in degrees.

This understanding is the basis of activities involving folding and tearing models of triangles and their interiors. It leads to the observation that the sum of the measures (in degrees) of the three angles of a triangle is 180. Measurements made on physical models are, at best, approximations, limited by the precision of the scale of the measuring instrument, by the thickness of the pencil point, by the eyesight of the observer, etc. In view of these facts we do not expect the pupil to find that the sum of the measures of the three angles of the particular model of the triangle he is using is exactly 180. The variation in sums found does not change the mathematical fact. This is just another illustration of the difference between a provable mathematical fact and observations made on representations of mathematical figures.

Some opportunity for deductive thinking (argument based on agreements and pre-established facts) is provided for the more astute pupil. Braintwister exercises may lead the pupil to recognize that we can find the sum of the measures of the angles of a polygon knowing the sum of the measures of the three angles of a triangle. We may draw selected lines connecting the vertices of a polygon, thus creating triangles. Exercises set up in the children's book then lead to the statement that the sum of the measures (in degrees) of the angles of a polygon is $(n - 2)\ 180$ where

n is the number of sides of the polygon
$n - 2$ is the number of triangles we formed
180 is the sum of the measures (in degrees) of the angles of each triangle formed.

It is not expected that this formula will be presented to the pupil but some may perceive this intuitively.

Triangles of special interest are characterized by the sizes of their angles. These include the equilateral triangle, each of whose angles has a size of 60°; the 45°, 45° and 90° triangle; and the 30°, 60°, 90° triangle. While it is true that two triangles are congruent if they have congruent corresponding sides, it is not necessarily true that two triangles are congruent if they have congruent corresponding angles. Such triangles have the same shape but not necessarily the same size; they are said to be similar.

The exploration on page 137 reviews the definition of the isosceles triangle while the exploration, page 141, Angles of an Isosceles Triangle, develops the property that there are at least two congruent angles in an isosceles triangle. Exercises 5-7 in this exploration help the children locate the congruent angles in relation to the congruent sides.

The exploration, Equilateral Triangles, page 145, gives a review of equilateral triangles. The exploration, Angles of an Equilateral Triangle, page 148, is concerned with the idea that equilateral triangles have three congruent angles.

Exercise Set 5, page 152 serves as an introduction to scalene triangles and the relationship of sides and angles in triangles. Pupils may need help in completing these exercises but it is expected they will work independently. The exploration of Scalene Triangles, page 154, leads to the understanding that the largest angle of a triangle lies opposite the longest side and the smallest angle lies opposite the shortest side.

All the explorations referred to in this section are in sufficient detail in the pupil text to provide a suitable development for the teacher to follow.

Chapter 3

SIDE AND ANGLE RELATIONSHIPS OF TRIANGLES

ISOSCELES TRIANGLES

Exploration

What do we call a triangle which has at least two congruent sides? You have seen many of these triangles and no doubt are able to give them their correct name--isosceles triangles.

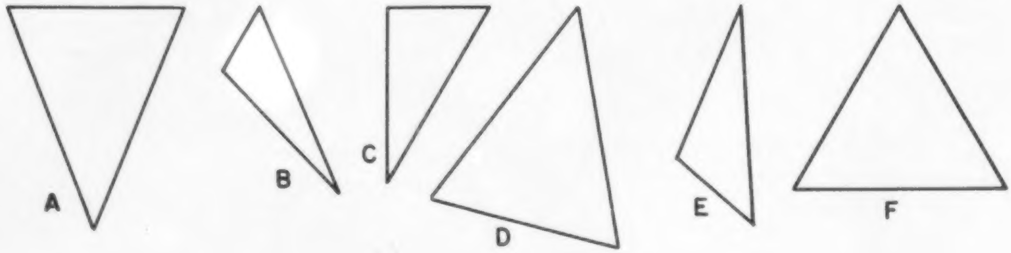
1. Make a model of a triangle using the strips and fasteners. For two sides, choose two strips of the same length, the longest ones you have. For the third side, choose a strip about half as long as the others you used.

Recall that a triangle with at least two congruent sides is called an isosceles triangle.

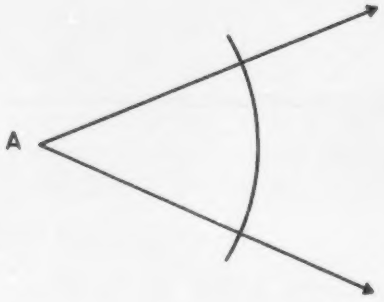
2. Draw an isosceles triangle in which the congruent sides are each two inches long and the third side is three inches long. Do you need to use your compass? (yes)

3. Which of these are isosceles triangles? (Triangles A, D, and F)

   How did you decide? (See whether at least two sides are congruent by using tracings or compass.)



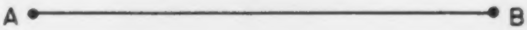
4. Use your compass and straightedge to copy this figure. In the figure, place the letters B and C at the intersection of the arc and rays. What segments are congruent? ($\overline{AB} \cong \overline{AC}$) Does this figure suggest a method for making an isosceles triangle?



Exercise Set 1

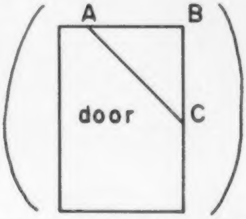
1. Use the method of Exploration Exercise 4 to draw an isosceles triangle, starting with a figure like this. How many such triangles could you draw?
(an unlimited number)

2.



Draw a line segment congruent to $\overline{AB}$. Name your line segment, $\overline{CD}$. How many isosceles triangles can you make on $\overline{CD}$ if it is not one of the congruent sides? *(As many as you please)* **Draw several of them. Must all these triangles lie on the same side of $\overline{CD}$?** *(no)*

3. Can you see any examples of isosceles triangles in our room? Could you, by drawing one line on the door, show an isosceles triangle? Do you see any other ways of making isosceles triangles in our room by drawing just one line?



Make $\overline{BA} \cong \overline{BC}$.

4. As you go home tonight look closely at things around you to see if you can find any examples of isosceles triangles. You may find some good examples in your neighborhood, at the dinner table, or even in your car. Most magazines have some good pictures of isosceles triangles in them, too.

BRAINTWISTERS

5. Draw two line segments of different lengths. Name one of them $\overline{EF}$ and the other one $\overline{GH}$. Now draw an isosceles triangle with two sides congruent to $\overline{EF}$ and the third side congruent to $\overline{GH}$. Is it easier if you draw the third side first? (Yes) Did you have any trouble drawing the isosceles triangle? (No, unless the $m\ \overline{GH} > m\ \overline{EF} + m\ \overline{EF}$.)

6. See if you can stump your teacher. Ask her to work problem 5 after you have marked $\overline{EF}$ and $\overline{GH}$ for her. Be sure to choose lengths so that she cannot draw the isosceles triangle. How did you do this? (Made $m\ \overline{GH} > m\ \overline{EF} + m\ \overline{EF}$.)

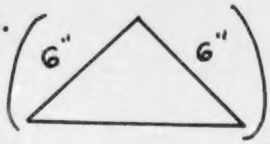
P141

ANGLES OF AN ISOSCELES TRIANGLE

Exploration

You have done many things with isosceles triangles. Let's look at them even more closely.

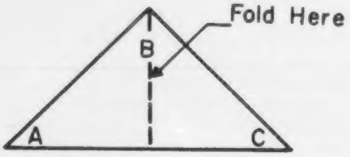
1. Draw an isosceles triangle with the congruent sides 6 inches long. Make the third side any length you choose. Do you have to be careful of the length you choose? (yes) Could you choose 12 inches for the length of the third side? (no) Why? (Because sides of length 6 inches, 6 inches, and 12 inches will not make a triangle)

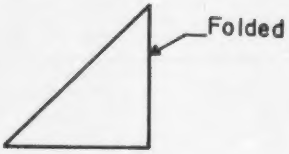


Could you choose a length greater than 12 inches for the length of the third side? (no) Why? (Because the sum of the measures of any two sides has to be greater than the measure of the third side.)

2. Cut out your isosceles triangle with its interior. Label the vertices so that $\overline{AB} \cong \overline{BC}$. Now fold it through point B so that side $\overline{BC}$ fits on side $\overline{BA}$.

3. These sketches show what you did.





Where did vertex C fall? (on vertex A) Is $\angle C$ congruent to $\angle A$? (yes) If you made the isosceles triangle carefully, $\angle C$ should fit exactly over $\angle A$ so that $\angle C \cong \angle A$.

4. Construct a triangle DEF with $\overline{DE} \cong \overline{FE}$. Trace $\triangle DEF$ on a sheet of thin paper and label the vertices.

Turn the sheet over and place

vertex D on F,
vertex E on E,
vertex F on D.

Is $\triangle DEF \cong \triangle FED$? (yes) If so, which angles are congruent? ($\angle D \cong \angle F$)

5. We call $\angle D$ the angle opposite $\overline{FE}$ since $\overline{FE}$ joins points on the sides of $\angle D$. In a similar manner, we call $\angle E$ the angle opposite $\overline{DF}$, and $\angle F$ the angle opposite $\overline{ED}$.

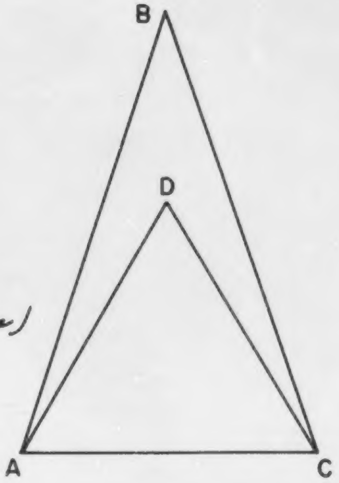
6. In isosceles $\triangle DEF$, which are the congruent sides? ($\overline{DE}$ and $\overline{FE}$) Are the congruent angles opposite the congruent sides? (yes)

7. Can you finish this sentence?
It seems to be true that if two sides of a triangle are congruent, then the angles (opposite these sides are congruent.)

Summary

In each isosceles triangle with which you worked you found at least two congruent angles. Every isosceles triangle has at least two congruent sides and at least two congruent angles, and the congruent angles are opposite the congruent sides.

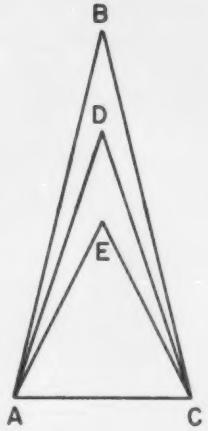
Exercise Set 2



1. In this drawing, $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. What kind of triangles are $\triangle ABC$ and $\triangle ADC$? (isosceles triangles)

   Which angle has the greater measure, $\angle BAC$ or $\angle DAC$? ($\angle BAC$)

   Is $\angle DAC \cong \angle DCA$? (yes)

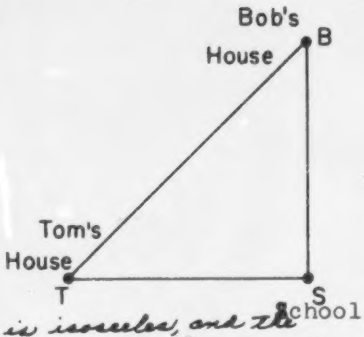


2. How many triangles can you find in this drawing? (3, they are $\triangle ABC$ and $\triangle ADC$ and $\triangle AEC$.)

   If $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{CD}$, and $\overline{AE} \cong \overline{CE}$, what kind of triangles are they? (isosceles triangles)

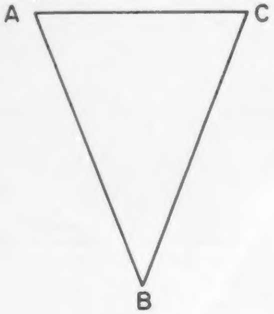
   If $\angle ACE \cong \angle CAE$ and $\angle DCA \cong \angle CAD$, are all four angles congruent? (no, because E is in the interior of $\angle CAD$ and also in the interior of $\angle DCA$.)

3. This drawing shows the location of Tom's house, Bob's house and their school. Bob and Tom live the same distance from school. Why is the angle at Tom's house congruent to the one at Bob's house? (The triangle is isosceles, and the angles at Tom's house and at Bob's house are opposite the congruent sides of the triangle.)



Suggestion: If Bob's house is directly north of the school and Tom's house is directly west of the school, what do you know about $\angle TSB$. ($\angle TSB$ is a right angle.)

4. In $\triangle ABC$, $\overline{AB} \cong \overline{CB}$. Choose any way you want to show that $\angle BAC \cong \angle BCA$.



(This conclusion can be reached by tracing, by folding, or by using properties of the isosceles triangle stated on page 142.)

EQUILATERAL TRIANGLES

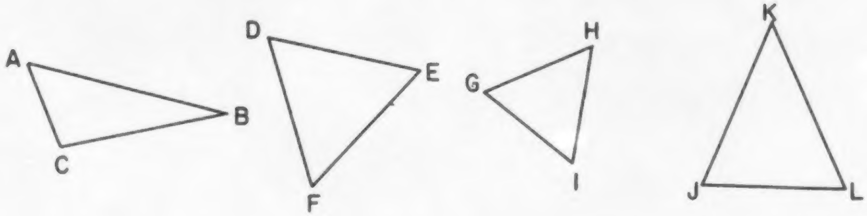
Exploration

A triangle which has three sides congruent to each other is called an equilateral triangle.

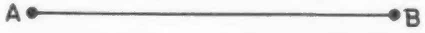
Make an equilateral triangle by using strips. Choose any strips you want, but you must be careful about one thing. What is it? (all the strips must be the same length because an equilateral triangle has three congruent sides.)

Exercise Set 3

1. Which of these are pictures of equilateral triangles? (△DEF, △GHI)

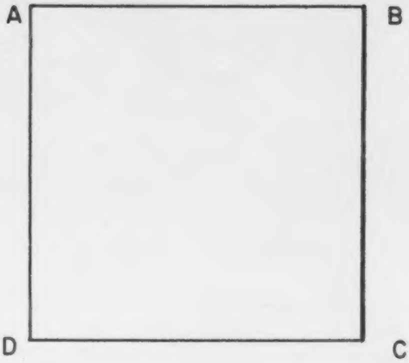


2. Draw an equilateral triangle using your compass and straightedge. Use the length of $\overline{AB}$ as the length of a side.



3. Draw three equilateral triangles. Make the first one with sides $1\frac{1}{2}$ inches in length, the second with sides $2\frac{1}{2}$ inches in length, and the third with sides $3\frac{1}{2}$ inches in length.

4. Figure ABCD is a square. What do you know about the sides of a square?
(all four sides are congruent)

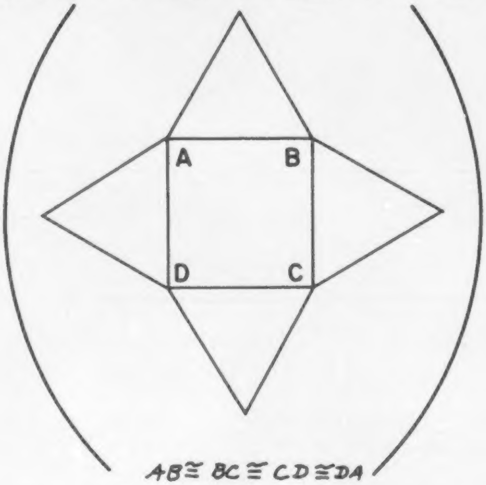


Name the congruent segments.
($\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$)

Trace figure ABCD on your paper.

Draw an equilateral triangle, using $\overline{AB}$ as one side. Draw the triangle so that its interior lies in the exterior of ABCD.

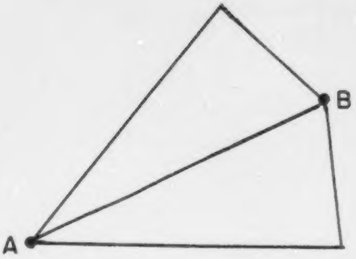
Draw equilateral triangles on $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ so that the interior of each triangle lies in the exterior of ABCD.



$AB \cong BC \cong CD \cong DA$

Is the triangle with side $\overline{AB}$ congruent to any of the other triangles you have just marked? (Yes, to each of the other three)

5. How many equilateral triangles can you draw which have a vertex at point A and another vertex at point B? (two) Draw them.



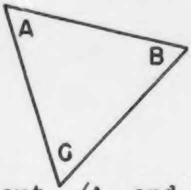
6. How many equilateral triangles with a side 2 inches long can you draw which will have one vertex at point B? (an unlimited number) How many isosceles triangles can you draw which will have a vertex at point A and a vertex at point B? (an unlimited number)

ANGLES OF AN EQUILATERAL TRIANGLE

Exploration

You know that an equilateral triangle has three congruent sides. Is there anything else that you think might be true about it? (the answers will vary.)

1. Draw an equilateral triangle which has a side 4 inches long. Name it ΔABC, putting the letters on the interior like this:



What must be true about $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$? ($\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{AC}$, $\overline{AC} \cong \overline{AB}$)

2. Since $\overline{AB} \cong \overline{BC}$, what can you say about $\angle A$ and $\angle C$?
($\angle A \cong \angle C$)

Since $\overline{BC} \cong \overline{AC}$, what can you say about $\angle A$ and $\angle B$?
($\angle A \cong \angle B$)

Now what can you say about $\angle A$, $\angle B$, and $\angle C$?
($\angle A \cong \angle B$, $\angle B \cong \angle C$, $\angle C \cong \angle A$)

3. Cut out triangle ABC with its interior. Fold it through point B so that $\overline{BC}$ falls on $\overline{BA}$. Does $\angle C$ fit exactly on $\angle A$? (yes) Now open up the figure and fold it through point C so that $\overline{BC}$ falls on $\overline{AC}$. Does $\angle B$ fit exactly on $\angle A$? (yes) Open the figure and fold it through A, so $\overline{AC}$ falls on $\overline{AB}$. Does $\angle C$ fit exactly on $\angle B$? (yes)

Exercise Set 4

1. Use your strips to make an equilateral triangle. Make another equilateral triangle that is congruent to the first one. Label the first triangle GHI and the second JKL. Can you place △JKL exactly on △GHI so that vertex J falls on vertex G, K falls on H, and L falls on I? (yes)

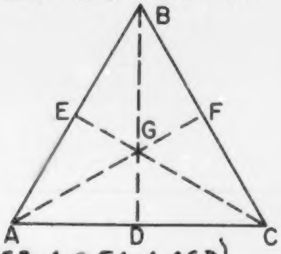
   Are there other ways in which △JKL will fit exactly on, that is, be congruent to △GHI?

(△JKL ≅ △GHI, △KLJ ≅ △GHI, △KJL ≅ △GHI, △LJK ≅ △GHI, △LKJ ≅ △GHI)

2. Draw an equilateral triangle with a side whose length is $1\frac{3}{4}$ inches. Are all three angles congruent? (yes)

3. Draw an equilateral triangle with a side whose length is $2\frac{1}{2}$ inches. Are the three angles congruent? (yes)

4. When you folded your equilateral triangle, in the Exploration, the folds made lines on it as in the drawing. Notice that we have named the folds $\overline{AF}$, $\overline{BD}$, and $\overline{CE}$. Name all of the different triangles you can find in the figure. Name each triangle only once. Can you find 16 triangles?



(△ABC, △ABD, △CBD, △AFC, △AFB, △CEB, △CEA, △AGD)
(△DGC, △CGF, △GFB, △BGE, △GEA, △AGC, △CGB, △AGB)

BRAINTWISTER

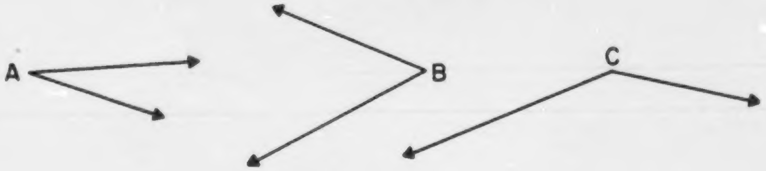
5. Draw an equilateral triangle ABC with a side of length 3 inches. Draw another equilateral triangle DEF with a side 4 inches long and another equilateral triangle with a side 5 inches long. Are these three triangles congruent? (no) Are the angles in all three equilateral triangles congruent? (yes)

BRAINTWISTER

6. If you drew an equilateral triangle with sides 8 inches long, would the angles be congruent to those of an equilateral triangle with sides 4 inches long? (yes) Would you expect this? (yes)

BRAINTWISTER

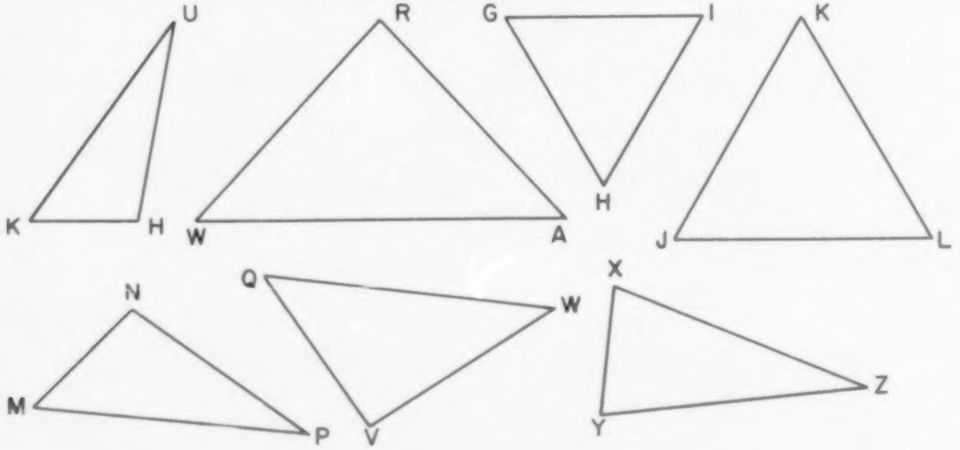
7. Can you draw an equilateral triangle with its angles congruent to these three angles?
(No, an equilateral triangle has three congruent angles)



Exercise Set 5

1. Divide the triangles below into four sets:

   Set A - Triangles with exactly two congruent sides
   {ΔWRA, ΔXYZ}
   Set B - Triangles with three congruent sides
   {ΔGHI, ΔJKL}
   Set C - All triangles not in Set A or Set B
   {ΔKHU, ΔMNP, ΔQVW}
   Set D - All triangles with at least two congruent sides
   (Set D = Set A $\cup$ Set B)



2. All the triangles in Set A or in Set B are (isosceles) triangles.

   All the triangles in Set B are (equilateral) triangles.

   All the triangles in Set A are (isosceles) triangles.

3. What can you say about the triangles in Set C?
   (Triangles which have no two sides congruent are called scalene triangles.)

4. Divide the triangles above into four new sets.

   Set E - Triangles with exactly two congruent angles. (Set E = Set A)
   (If you need to, trace the angles on
   paper to help you decide.)

   Set F - Triangles with three congruent angles. (Set F = Set B)

   Set G - Triangles with no two angles congruent. (Set G = set C)

   Set H - Triangles with at least two angles congruent.
   (Set H = Set D)

5. Look at the eight sets you have listed. Which sets have exactly the same members? (See answers to exercise 4)

   Which sets in Exercise 1 are subsets of other sets?
   (Set A is a subset of Set D. Set B is a subset of Set D. Each set is a subset of itself.)

6. What is the intersection of Sets A and B? (Set B, or $\{\Delta GHI, \Delta JKL\}$)

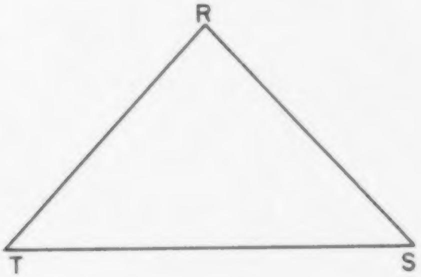
   What is the intersection of Sets G and B? (The empty set)

   What is the union of Sets H and C? $\{\Delta KHU, \Delta WRA, GHI, \Delta JKL, \Delta MNP, \Delta QVW, \Delta XYZ\}$

SCALENE TRIANGLES

Exploration

You know that in a triangle the angles opposite two congruent sides are also congruent. What do you think about angles opposite sides of unequal length?



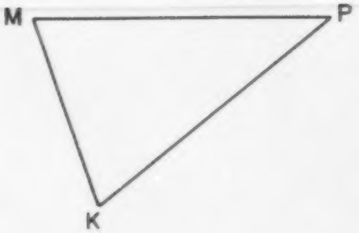
1. In $\triangle RST$, $\overline{RS} \cong \overline{RT}$, so which angles must be congruent? $(\angle T \cong \angle S)$

2. Which is longer, $\overline{RS}$ or $\overline{ST}$? $(\overline{ST})$

3. What angle is opposite $\overline{RS}$? $(\angle T)$
   What angle is opposite $\overline{ST}$? $(\angle R)$

4. **By tracing on thin paper compare $\angle R$ with $\angle T$.**
   Which angle has the greater measure? $(\angle R)$

5. Which side is opposite the **greater angle,** $\overline{RS}$ or $\overline{ST}$? $(\overline{ST})$



6. In triangle MPK choose a pair of sides which are not congruent. ($\overline{MP}$ and $\overline{MK}$ or $\overline{PK}$ and $\overline{MK}$.)

7. Tell which angle is opposite each of the sides you chose. ($\angle K$ is opposite to $\overline{MP}$. $\angle M$ is opposite $\overline{PK}$
$\angle P$ is opposite to $\overline{MK}$.)

8. Compare the sizes of the two angles.
($\angle K$ is larger than $\angle P$. $\angle M$ is larger than $\angle P$.)

9. Arrange your answers for Exercises 6, 7, 8 like this:

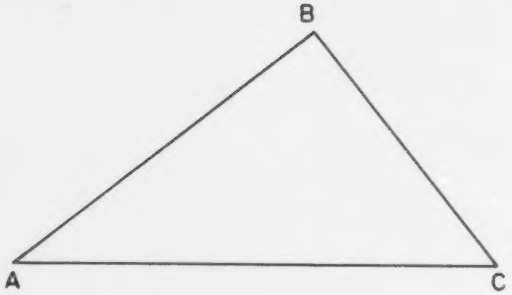
Longer side: ($\overline{MP}$ or $\overline{PK}$) Shorter side: ($\overline{MK}$)

Angle opposite longer side: ($\angle K$ or $\angle M$)

Angle opposite shorter side: ($\angle P$)

Larger angle: ($\angle K$ or $\angle M$) Smaller angle: ($\angle P$)

10. Is the larger angle opposite the longer side or the shorter side? (Larger angle is opposite the longer side.)



11. Are any two sides of $\triangle ABC$ congruent? (no) What is this kind of triangle called? (scalene)

12. Arrange the sides in order of length. Then name the angle opposite each side.

Longest side: ($\overline{AC}$) Opposite angle: ($\angle B$)
Next longest side: ($\overline{AB}$) Opposite angle: ($\angle C$)
Shortest side: ($\overline{BC}$) Opposite angle: ($\angle A$)

13. Compare the sizes of the angles:

Largest size angle: ($\angle B$)
Next largest angle: ($\angle C$)
Smallest angle: ($\angle A$)

14. What do you notice about your answers to Exercises 12 and 13?

(The largest angle is opposite the longest side.
The next largest angle is opposite the next longest side.
The smallest angle is opposite the shortest side.)

Exercise Set 6

1. Draw a scalene triangle. Name it with letters.

2. List the sides in order of size from longest to shortest.

3. List the angles in order of size from largest to smallest.

4. Which length side is opposite the largest angle?
(The longest side is opposite the largest angle.)
Which length side is opposite the smallest angle?
(The shortest side is opposite the smallest angle.)

5. Can you make a triangle out of three sticks whose lengths are 5 inches, 7 inches, and 9 inches? (yes) If so, where will the greatest angle be?
(The largest angle will be opposite the side whose length is nine inches.)

6. What kind of triangle can you make out of three sticks of length 8 inches, 8 inches, and 6 inches? (isosceles) What should be true about the sizes of the angles of the triangle? (Angles opposite the 8 inch sides will be congruent. The smallest angle will be opposite the 6 inch side.)

MEASURING ANGLES OF A TRIANGLE

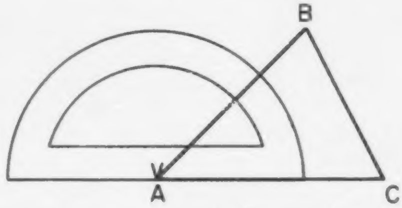
Objective: To develop the following understandings and skills.

1. (a) the V point of the protractor must be placed on the vertex of the angle
   (b) the zero ray of the protractor may be placed on either ray of the angle
   (c) the sides of the triangle on which the rays of the angle lie may have to be extended

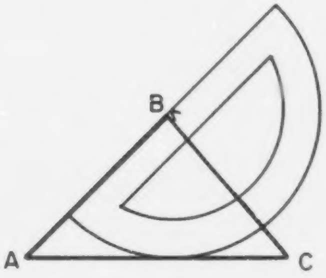
Materials Needed:

Teacher - Straightedge, chalkboard protractor
Pupil - Straightedge, protractor

Vocabulary: No new words in this section



To check the measure of $\angle A$, the pupil places the zero ray of the protractor on $\overline{AC}$. At this time it is well to stress the point that V must be on the vertex of the angle being measured as above, so that the error pictured below does not occur.

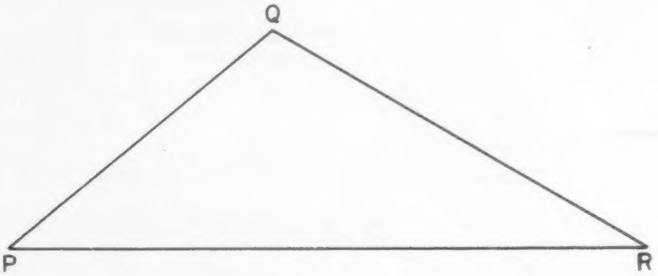


(The V of the protractor is not on the vertex of angle A, the angle being measured.)

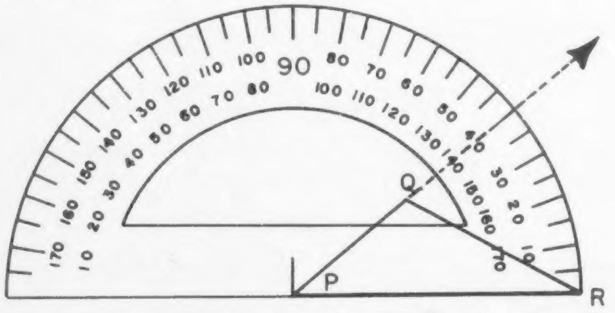
ANGLES OF A TRIANGLE

Exploration

You have learned how to measure an angle with your protractor. Now we will measure angles which are determined by triangles. Suppose we wish to measure the angles of △PQR.

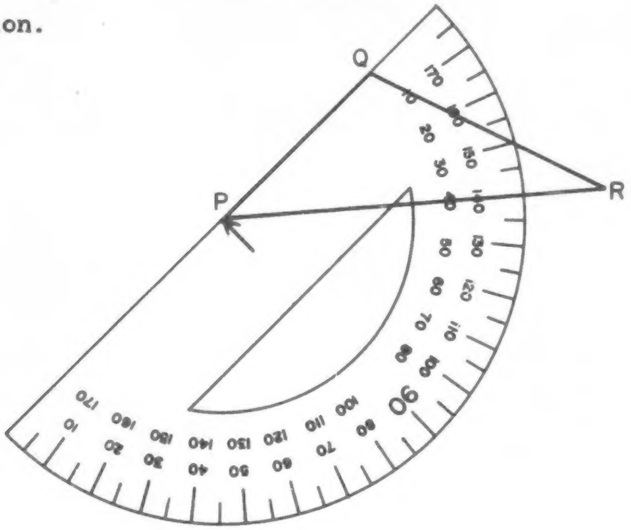


1. Where will you place the point V of your protractor to measure ∠QPR? (Place point V of the protractor on vertex P)
If you place the point V of your protractor on P, along which ray of ∠P may the zero ray of your protractor be placed? (The zero ray of the protractor may be placed along $\overline{PR}$ or $\overline{PQ}$)
2. **If you place the zero ray along $\overline{PR}$, which side of ∠P should you extend, if necessary?** ($\overline{PQ}$)
A picture of the protractor placed to measure ∠P would look like this:



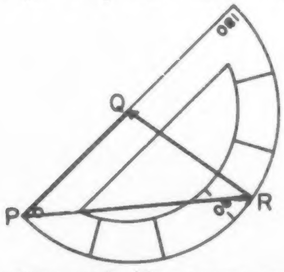
What is the measure (in degrees) of ∠P? (m∠P=40)

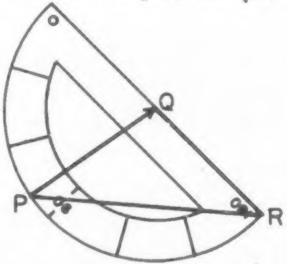
3. **Now place the protractor so the point V of your protractor is on P and the zero ray is on $\overline{PQ}$. This is a picture of the protractor in this position.**


Q
P
R
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170

To find the measure of ∠P, should you look at $\overline{QR}$ or $\overline{PR}$? ($\overline{PR}$) Do you get the same reading on the protractor for the m∠P as before? (m ∠P = 40)

4. On which vertex of △PQR must we place point V of the protractor to measure ∠PQR? (vertex Q) Along which sides of △PQR may the zero ray of the protractor be placed? ($\overline{PQ}$ or $\overline{QR}$)





Do you get the same measure of ∠Q both times? (yes)

5. Measure $\angle R$ by placing the zero ray of the protractor along one of the rays of the angle. Repeat by using the other ray of the angle. Are the two measures the same?
(yes, $m\angle R = 30$)

Summary

To measure an angle of a triangle, the point V of the protractor must be placed on the vertex of the angle. The zero ray may be placed along either ray of the angle. The sides of the triangle on which the rays of the angles lie may have to be extended.

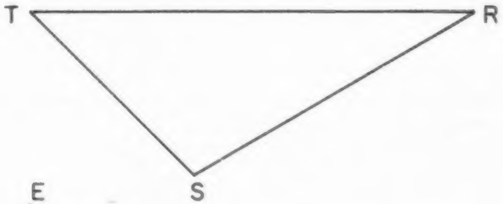
Exercise Set 7

Use your protractor to measure the 3 angles of each triangle. Check your measure (in degrees) in each case by placing the zero ray of the protractor along the other ray of the angle.

1. m∠T = (45)

   m∠S = (105)

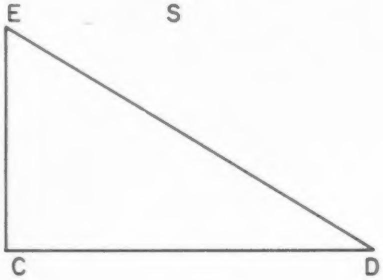
   m∠R = (30)



2. m∠C = (90)

   m∠D = (32)

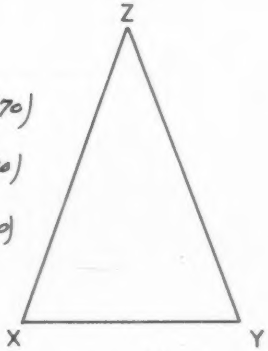
   m∠E = (58)



3. m∠X = (70)

   m∠Y = (70)

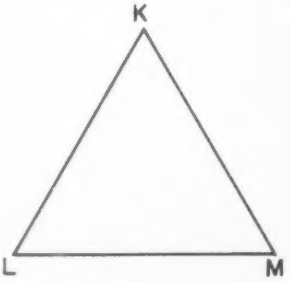
   m∠Z = (40)



4. m∠K = (60)

   m∠M = (60)

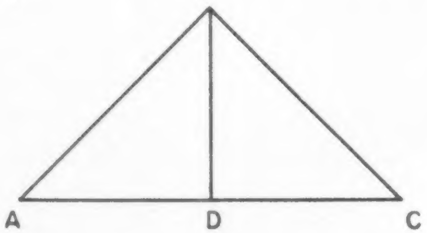
   m∠L = (60)



5. m∠BAD = (45)

   m∠ABD = (45)

   m∠BDA = (90)



SUM OF MEASURES OF ANGLES OF A TRIANGLE

Objective: To develop the following understandings and skills:

1. The sum of the measures, in degrees, of the three angles of a triangle is 180.
2. The sum of the measures, in degrees, of the angles of a quadrilateral is 2 × 180.
3. The sum of the measures, in degrees, of the angles of a polygon of n sides is $(n - 2) \times 180$.

Materials Needed:

Teacher - Straightedge, chalkboard protractor, 2 demonstration models of triangles and their interiors, flannel board

Pupil - Straightedge, compass, ruler, scissors

Vocabulary: **No new words in this section**

Octons--introduced in the fifth grade chapter, Measurement of Angles.

This exploration lends itself well to teacher demonstration. The two experiments which lead to the conclusion that the sum of the measures (in degrees) of the angles of a triangle is 180, both depend upon the understanding developed in the section Sum of the Measures of Angles. The experiment involving tearing the triangular region and placing the angles so that their interiors (if all points in their interiors are considered) fill the half plane is efficiently done using a flannel board.

Exercise Set 8 puts to use the conclusion reached in the Exploration and provides arithmetic application in the first exercise. Exercises 2-5 are developmental and intended to give the pupil an opportunity to do some very simple deduction in establishing that the sum of the measures of the four angles of a quadrilateral is twice that of a triangle, 360. The answers to exercises 6-9 are intended to be reached by deduction (because the pupil knows the sum of the measures of the angles of a triangle) and not by actual measure. There should be class discussions following the completion of the exercises.

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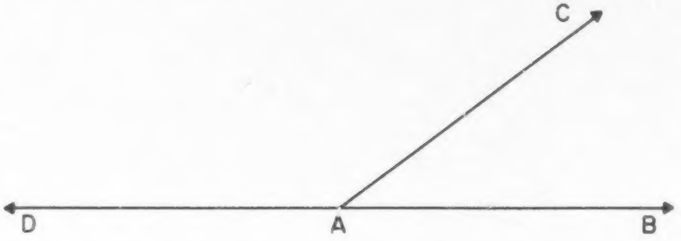
SUM OF MEASURES OF ANGLES OF A TRIANGLE

Exploration

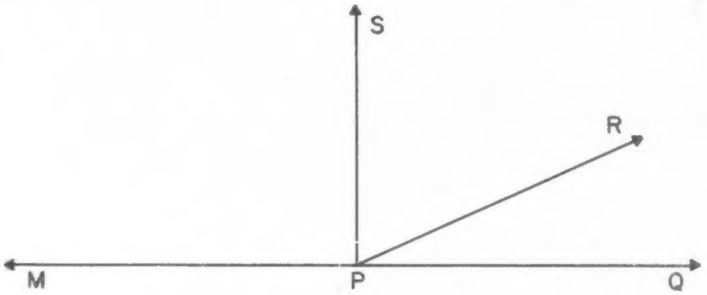
1.



If $\overrightarrow{AD}$ and $\overrightarrow{AB}$ are on a line, then, in degrees,
$m\angle DAC + m\angle BAC = \underline{180}$, and, in octons (see Grade 5 text, Chapter 6), $m\angle DAC + m\angle BAC = \underline{8}$

Check with your protractor.

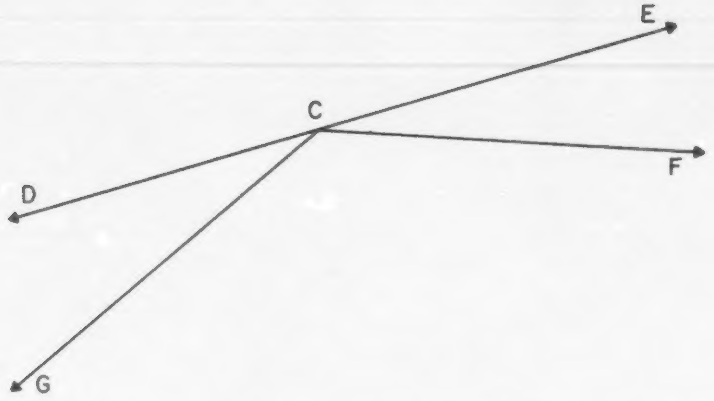
2.



If $\overleftrightarrow{MPQ}$ is a straight line, then in degrees
$m\angle MPS + m\angle SPR + m\angle RPQ = \underline{180}$ and in octons
$m\angle MPS + m\angle SPR + m\angle RPQ = \underline{8}$.

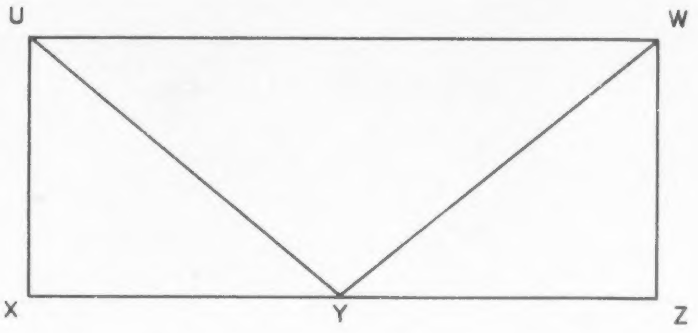
Check with your protractor.

3.



If $\overleftrightarrow{DCE}$ is a straight line, for which three angles is the sum of the measures (in degrees) exactly 180? Check with your protractor. ($\angle DCG$, $\angle GCF$, and $\angle FCE$)

4.

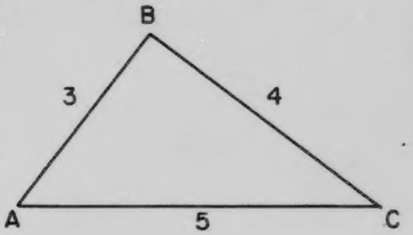


Use your protractor to see if the following statement seems reasonable.

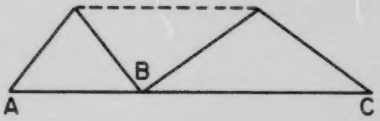
$m\angle UYX + m\angle UYW + m\angle WYZ = 180$

($m\angle UYX + m\angle UYW + m\angle WYZ = 180$ within the physical limitations on measurement.)

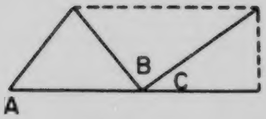
5. Draw a triangle whose sides will have measures, in inches, of 3, 4, and 5. Cut it out with its interior. Hold it like this:



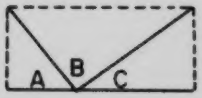
Mark the midpoint of $\overline{AB}$ and the midpoint of $\overline{BC}$. Fold $\triangle$ ABC through the midpoints; B should fall on $\overline{AC}$.



Then fold so that C falls on point B.



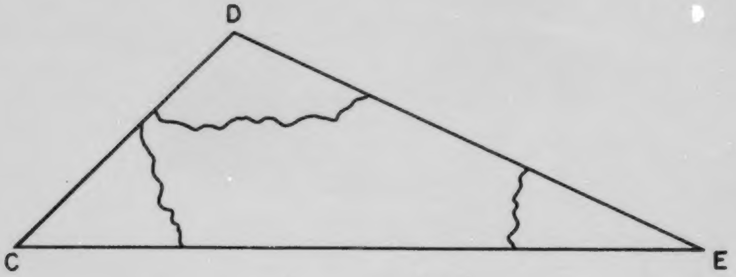
Then finally, A falls on B.



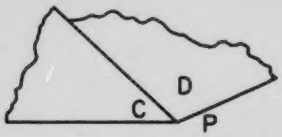
Complete the mathematical sentence

$$m\angle A + m\angle B + m\angle C = \underline{(180)}.$$

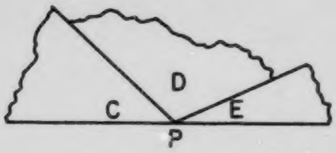
6. Cut out any triangle with its interior. Label each angle of the triangle. Tear the model like this. (Keep all the parts)



Place the models of $\angle C$ and $\angle D$ so that $\angle C$ and $\angle D$ have a common vertex, P, and a common side.



Then place $\angle E$ so that $\angle D$ and $\angle E$ have the same vertex, P, and a common side.



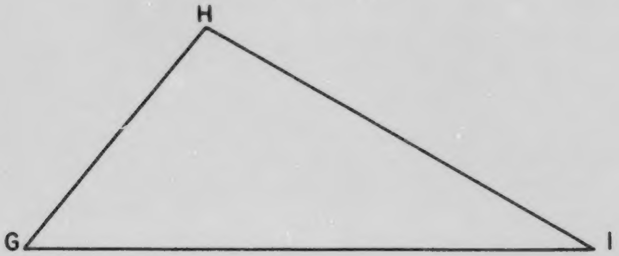
What do you observe?

Complete the sentence

$$m\angle C + m\angle D + m\angle E = (180)$$

7. In Exercise 5, $\angle A$, $\angle B$ and $\angle C$ were the three angles of a triangle. In Exercise 6, $\angle C$, $\angle D$ and $\angle E$ were the three angles of a triangle. In each case what did you find to be the sum of the measures in degrees of the three angles of a triangle? (180)

8.



What do you think is the sum of the measures of the three angles of $\triangle GHI$? (180)

Use your protractor to find the measures of the angles

$m\angle G =$ (50)

$m\angle H =$ (100)

$m\angle I =$ (30)

Add the measures. If your sum is not 180, can you account for this? (If the sum is not 180, it may be because it is difficult to measure angles exactly with the protractor. We showed in 3 ways that the sum of the measures, in degrees, of the three angles of a triangle is 180.)

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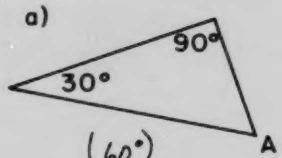
Exercise Set 8

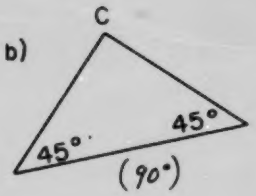
1. In each triangle below the sizes of certain angles are shown. Find the size of each angle whose vertex is named with a letter. Sides marked || are congruent segments.

a)



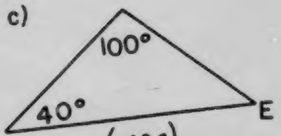
(60°)

b)



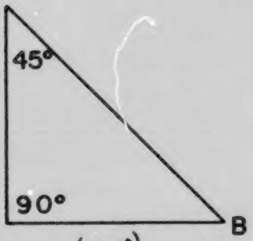
(90°)

c)



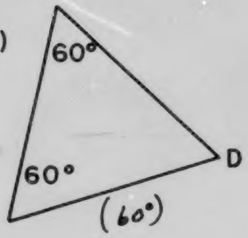
(40°)

d)



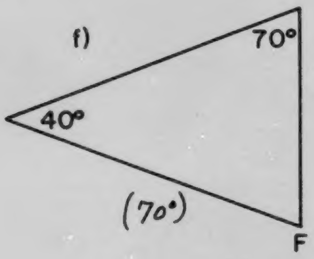
(45°)

e)



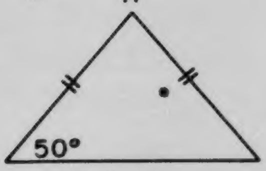
(60°)

f)



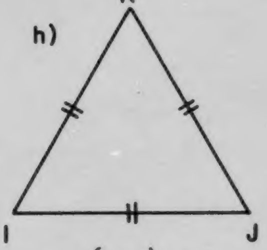
(70°)

g)



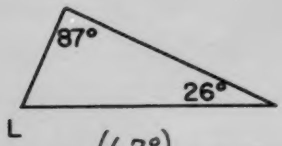
(80°)

h)



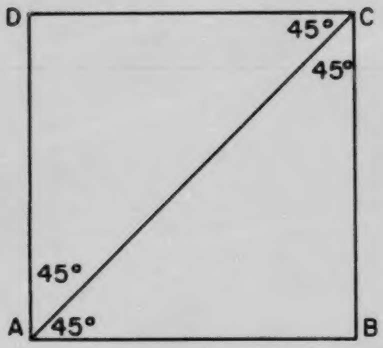
(60°)

i)



(67°)

2.



What is the measure (in degrees) of $\angle DAC$? (45)

What is $m\angle CAB$? (45)

What is $m\angle DAB$? (90)

Is $m\angle DAC + m\angle CAB = m\angle DAB$? (yes)

What is the measure of $\angle BCA$? (45)

What is $m\angle ACD$? (45)

What is $m\angle BCD$? (90)

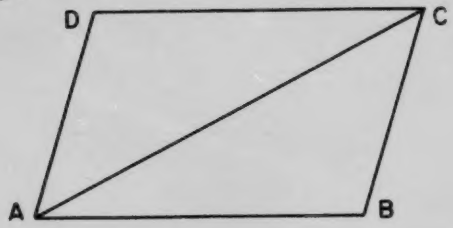
Is $m\angle BCA + m\angle ACD = m\angle BCD$? (yes)

What is $m\angle D$? (90) What is $m\angle B$? (90)

Complete the sentence:

$$m\angle DAB + m\angle B + m\angle BCD + m\angle D = \underline{(360)}$$

3.



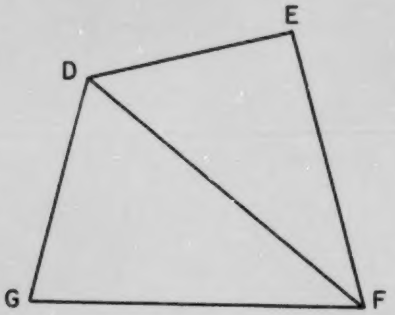
What is the sum of the measures of the three angles of ΔABC? (180) What is the sum of the measures of the three angles of ΔADC? (180)

Complete the sentence:

$$m\angle DAB + m\angle B + m\angle BCD + m\angle D = \underline{(360)}$$

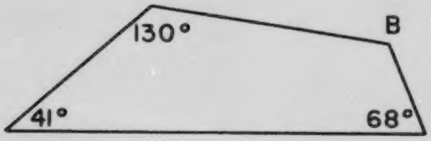
What is the sum of the measures of the 4 angles of the quadrilateral ABCD? (360)

4.



What is the sum of the measures of the angles of $\triangle$ GDF? (180) What is the sum of the measures of the angles of $\triangle$ DEF? (180) What is the sum of the measures of the angles of quadrilateral DEFG? (360)

5.



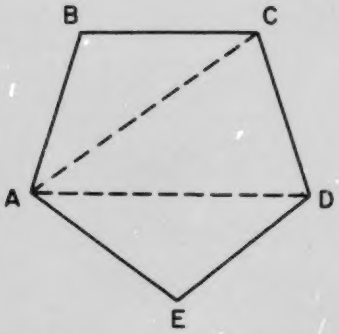
$m\angle B$ = (121)

How do you know?

(The sum of the measures, in degrees, of the angles of a quadrilateral is 360. This is true because any quadrilateral can be divided into two triangles. The sum of the measures of the angles of each triangle is 180.)

BRAINTWISTERS

6.

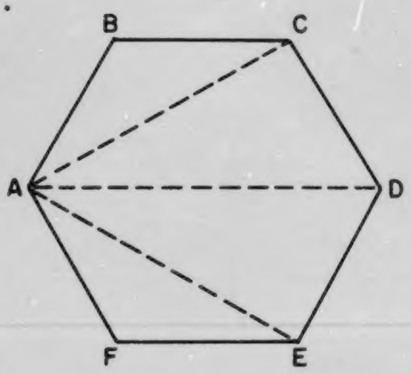


Complete the mathematical sentence:

$m\angle BAE + m\angle B + m\angle BCD + m\angle CDE + m\angle E$ = (540)

Hint: How many triangles have been made to fit in the interior of the figure ABCDE? (3)

7.



What is the sum of the measures of the angles of the figure ABCDEF? (720)

8. What is the sum of the measures of the angles of a polygon of 7 sides? (900. The figure has seven sides, so $(7-2) \times 180 = 900$)

9. What is the sum of the measures of the angles of a polygon of 202 sides? (36,000. The figure has 202 sides so $(202-2) \times 180 = 36{,}000$)

10. Name some numbers which are the sum of the measures (in degrees) of the angles of a polygon. (answers will vary)
    Name the six smallest such numbers. (180, 360, 540, 720, 900 and 1080)

11. Draw a polygon with more than 4 sides. Mark a point in its interior. Name the point, C. Draw line segments from C to each vertex of the polygon. In the interior of the polygon draw a small circle using C as a center. From your drawing tell why the sum of the measures (in degrees) of the angles of a polygon is $(n - 2) \times 180$ or $n \times 180 - 360$. (The sum of the measures of the angles of all the triangles includes the measures of the angles at C. The sum of the measures of the angles at C is 360. The angles at C should not be included. Hence we subtract 360 from $n \times 180$.)

12. Could the sum of the measures (in degrees) of the angles of a polygon be 250? Why? (No, since $250 \neq (n-2) \times 180$.)

13. Could the sum of the measures (in degrees) of the angles of a polygon be 160? Why? (No, since the sum of the measures (in degrees) of a polygon is at least 180, or, since $160 \neq (n-2) \times 180$.)

MEASURES OF ANGLES OF SPECIAL TRIANGLES

Objective: To develop the following understandings and skills:

1. **The measure (in degrees) of each angle of an equilateral triangle is 60.**
2. **To make an angle whose measure (in degrees) is 60, you may draw an equilateral triangle.**
3. Two triangles of special interest have angles whose sizes are 30°, 60°, 90° and 45°, 45°, 90°.
4. Two triangles which have 3 pairs of corresponding angles with the same measures are not necessarily congruent.

Materials Needed:

Teacher - Straightedge, protractor
Pupil - Straightedge, protractor, chalk

Vocabulary: No new words in this section

> This Exploration is also one which the teacher might allow a group of students to do independently. Several of the understandings of this section are developed in the Exercise, Set 9. Therefore it is essential that when the pupils complete this exercise they check their results with the teacher. If time is short this section might be included only for pupils with interest and ability for independent study.

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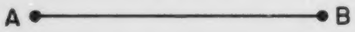
MEASURES OF ANGLES OF SPECIAL TRIANGLES

Exploration

You have studied angle relationships in some special triangles.

1. Use your compass to draw an isosceles triangle, with the length of each of the congruent sides the same as the length of $\overline{AB}$. How many such triangles can be drawn? (as many as you wish)



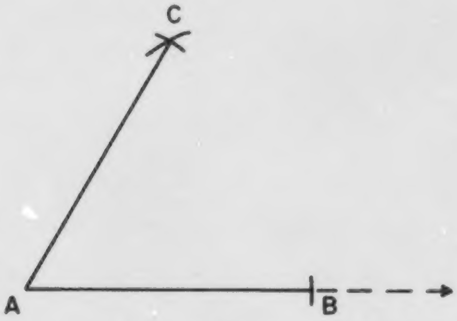
Measure the angles. What seems to be true about the angles of an isosceles triangle?
(An isosceles triangle has at least two angles which are congruent and thus have the same measure. An infinite number of isosceles triangles can be drawn with the length of the congruent sides the same as the length $\overline{AB}$. The measures of the angles would be equal.)

2. Did each one of your classmates find that the congruent angles of his isosceles triangle had the same measure? If the answer is "no," can you think of the reasons for this?
(Yes, two angles of any one isosceles triangle should have the same measure. If they do not, it may be because the isosceles triangle was not constructed exactly right or because the angle can not be measured exactly with the protractor.)

3. What did you learn about the angles of an equilateral triangle? How should their measures compare? Use your compass to draw an equilateral triangle. Choose any convenient length as the length of a side. Measure the angles. How do the measures compare?
(The angles of an equilateral triangle are congruent to each other, their measures should be equal.)

4. How many different equilateral triangles were made in answer to question 3? How did the measures of the angles of your triangle compare with the measures of the angles of the triangles of your classmates?
(Probably each pupil drew an equilateral triangle that was different from the other pupils equilateral triangles. All children should have 60 for the measure in degrees of each angle of their equilateral triangles.)

5. What number is the measure, in degrees, of every angle of an equilateral triangle? Using only compass and straightedge, draw an angle whose measure in degrees is 60. Did you draw an equilateral triangle? Did your drawing look like this?



Was it necessary to draw $\overline{CB}$ to complete the triangle in order to draw an angle whose measure is 60? Check your construction with your protractor.

(It is not necessary to draw $\overline{CB}$ to complete the triangle in order to draw an angle whose measure is 60.)

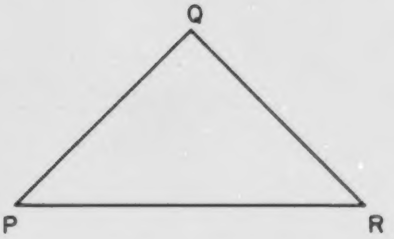
Summary

Angles opposite congruent sides of an isosceles triangle have the same measure.

Sixty is the measure (in degrees) of each angle of any equilateral triangle.

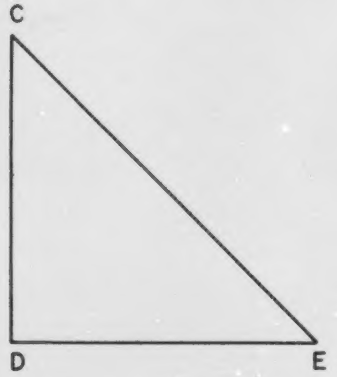
Exercise Set 9

1. $\triangle PQR$ is isosceles with $\overline{QP} \cong \overline{QR}$. Use your protractor to find $m\angle P$. What do you expect for the $m\angle R$? Check your answer by using the protractor. ($m\angle P = m\angle R$)



2. A triangle whose angles have measures (in degrees) of 45, 45, and 90 is of special interest because there are so many examples of it in your surroundings. What examples of right angles can you see in your classroom? Be ready to show how to complete a 45°, 45°, 90° triangle by drawing one segment.

3. In $\triangle CDE$, $\overline{CD} \cong \overline{DE}$. Which angles have the same measure?

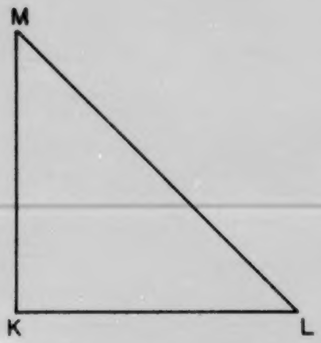


Use your protractor to find the following measures.

$m\angle D$ = (90)

$m\angle C$ = (45)

$m\angle E$ = (45)

4. 

△MKL is an isosceles triangle. $\overline{MK} \cong \overline{KL}$. Find

$m\angle K$ = (90)

$m\angle M$ = (45)

$m\angle L$ = (45)

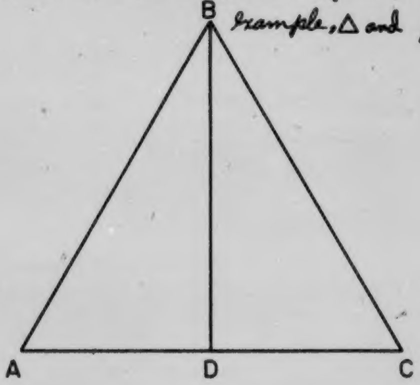
How do these measures compare with those of Exercise 3?

Are △ CDE and △ MKL congruent? (The measures of the angles are the same but △ CDE and △ MKL are not congruent.)

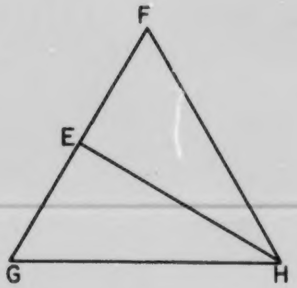
5. Have you seen other examples of triangles which were not congruent, and whose angles have the same measure? (All equilateral triangles have angles which are congruent but the triangles are not necessarily congruent.)

Tell whether this statement is true or false: Two triangles may have three pairs of corresponding angles with the same measures and still not be congruent. (true, for example, △ and △.)

6. How many triangles are there in this figure? (3) One of them is an equilateral triangle. **Which one?** (△ABC) **Name** 3 angles whose measure is 60. ($\angle BAC$, $\angle ACB$, $\angle ABC$)



7. $\triangle$ FGH is an equilateral triangle, and $\overline{EH}$ is drawn so that $\angle$HEF is a right angle. Look at $\triangle$ FEH. For which angles do you know the measures?
($m\angle F = 60$, $m\angle HEF = 90$, $m\angle FHE = 30$)



8. Look at $\triangle$ EGH. List the measures of the 3 angles of $\triangle$ EGH. ($m\angle G = 60$, $m\angle GEH = 90$, as $m\angle GHE = 30$)

9. Are all triangles whose angles have measures of 30, 60, and 90, congruent? (no)

   A triangle whose angles have measures of 30, 60, and 90 is another triangle of special interest. Look around you for models of right angles. Be ready to show how to draw a 30°, 60°, 90° triangle by drawing one segment.

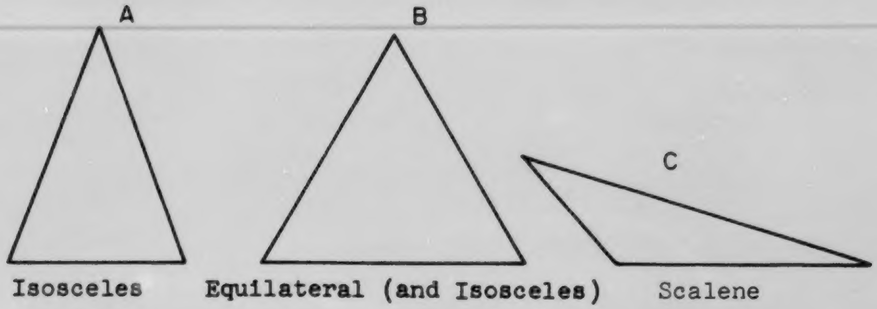
10. Complete the following statement:

    (a) **The measure (in degrees) of any angle of an equilateral triangle is** (60)**.**

    (b) **Two triangles of special interest have angle measures (in degrees) of** 45**,** 45**,** 90 **and** 30**,** 60**,** 90**.**

Suggested Test Items

1. Write A, B, or C in the space provided to show the triangle, or triangles, for which the sentence is true.



A + B (a) The triangle has at least two sides which are congruent to each other.

C (b) The triangle has no congruent angles.

B (c) The triangle has three congruent angles.

A + B (d) The triangle has at least two angles which are congruent to each other.

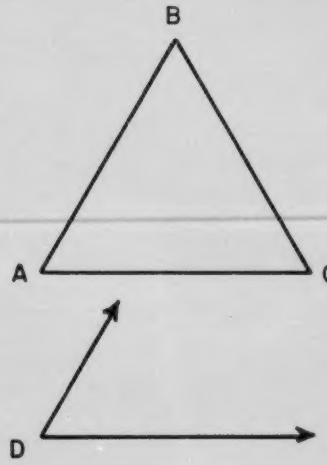
B (e) The triangle has three sides which are congruent to each other.

C (f) The triangle has no congruent sides.

A (g) The triangle has exactly two sides which are congruent to each other.

A (h) The triangle has exactly two angles which are congruent to each other.

2. $\triangle$ ABC is an equilateral triangle. If $\angle$A and $\angle$D are both the same size, how do you expect $\angle$C to compare in size with $\angle$D? Tell what facts you learned to make you think your answer is correct. $\angle C \cong \angle D$



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Practice Exercises

Find the number that y represents. Express all answers in simplest form.

1. a) $\frac{1}{2} + \frac{1}{8} = y$ ($\frac{5}{8}$)
   b) $8\frac{1}{4} + 7\frac{1}{4} = y$ ($15\frac{1}{2}$)
   c) $5\frac{1}{8} + 6\frac{5}{8} = y$ ($11\frac{3}{4}$)
   d) $\frac{1}{2} + \frac{3}{4} = y$ ($1\frac{1}{4}$)
   e) $7\frac{1}{2} + 5\frac{3}{4} = y$ ($13\frac{1}{4}$)
   f) $\frac{2}{3} + \frac{1}{3} = y$ (1)
   g) $\frac{3}{4} + \frac{1}{8} = y$ ($\frac{7}{8}$)
   h) $6\frac{3}{4} + 5\frac{1}{8} = y$ ($11\frac{7}{8}$)

2. a) $36.3 + 23.0 = y$ (59.3)
   b) $5.46 + 1.48 = y$ (6.94)
   c) $269 + 287 = y$ (556)
   d) $881 + 482 + 886 = y$ (2249)
   e) $876 + 867 + 972 + 659 = y$ (3374)
   f) $993 + 364 + 848 = y$ (2205)
   g) $8.98 + 4.07 = y$ (13.05)
   h) $698 + 589 = y$ (1287)

3. a) $0.3 \times 0.3 = y$ (.09)
   b) $1.1 \times 2 = y$ (2.2)
   c) $64.1 \times 9 = y$ (576.9)
   d) $6.3 \times 8 = y$ (50.4)
   e) $92 \times 7.309 = y$ (672.428)
   f) $5 \times 0.9 = y$ (4.5)
   g) $10 \times 3.6 = y$ (36)
   h) $10 \times 8.54 = y$ (85.4)

4. a) $\frac{2}{3} - y = \frac{7}{15}$ ($\frac{1}{5}$)
   b) $y - \frac{2}{3} = 8\frac{1}{2}$ ($9\frac{1}{6}$)
   c) $4\frac{1}{3} - 3\frac{1}{4} = y$ ($1\frac{1}{12}$)
   d) $1\frac{3}{4} - y = \frac{11}{12}$ ($\frac{5}{6}$)
   e) $y - 5\frac{3}{5} = 2\frac{9}{10}$ ($8\frac{1}{2}$)
   f) $2\frac{1}{4} - 1\frac{3}{8} = y$ ($\frac{7}{8}$)
   g) $\frac{3}{4} - \frac{1}{2} = y$ ($\frac{1}{4}$)
   h) $y - 2\frac{3}{4} = 3\frac{1}{2}$ ($6\frac{1}{4}$)

5) a) $\frac{1}{2} \times \frac{1}{6} = y$ ($\frac{1}{12}$)
   b) $\frac{9}{10}$ of $\frac{4}{9} = y$ ($\frac{2}{5}$)
   c) $5 \times \frac{1}{8} = y$ ($\frac{5}{8}$)
   d) $\frac{16}{24} \times \frac{3}{8} = y$ ($\frac{1}{4}$)
   e) $\frac{2}{10} \times \frac{6}{10^2} = y$ ($\frac{12}{10^3}$)
   f) $2\frac{1}{2} \times 6 = y$ (15)
   g) $47 \times 5\frac{1}{5} = y$ ($244\frac{2}{5}$)
   h) $\frac{3}{4}$ of $36 = y$ (27)

6. a) 30 × 478 = y (14,340)
   b) 256 × 100 × 2 = y (51,200)
   c) (40 × 984) + (6 × 984) = y (45,264)
   d) (7,000 × 875) + (200 × 875) + (5 × 875) = y (6,304,375)
   e) 106 × 470 = y (49,820)
   f) (300 × 9,150) + (80 × 9,150) = y (3,477,000)
   g) 400 × (600 + 52) = y (260,800)
   h) 209 × 639 = y (133,551)

7. a) (2,200 + 99) ÷ 11 = y (209)
   b) 69,360 ÷ y = 17 (4,080)
   c) 3,332 ÷ 49 = y (68)
   d) 4,984 ÷ 56 = y (89)
   e) 331,705 ÷ y = 407 (815)
   f) (217,200 + 33,304) ÷ 724 = y (346)
   g) 50,542 ÷ 683 = y (74)
   h) 546,984 ÷ 642 = y (852)

8. a) $8\frac{1}{6} = 7\frac{y}{18}$ (21)
   b) $7\frac{2}{3} = 6\frac{y}{9}$ (15)
   c) $\frac{4}{12} = \frac{y}{36}$ (12)
   d) $\frac{y}{20} = \frac{2}{5}$ (8)
   e) $1\frac{3}{4} = \frac{y}{8}$ (14)
   f) $\frac{y}{21} = \frac{2}{7}$ (6)
   g) $\frac{1}{4} = \frac{12}{y}$ (48)
   h) $2\frac{1}{5} = \frac{y}{10}$ (22)

9. a) $3^2 \times y = 72$ (8)
   b) $y + 7^2 = 62$ (13)
   c) $81 \div 3^2 = y$ (9)
   d) $60 = y + 6^2 + 4^2$ (8)
   e) $4^3 \times y = 512$ (8)
   f) $43 = y + 2^4$ (27)
   g) $y \div 5^2 = 4$ (100)
   h) $\frac{3}{10^1} \times \frac{7}{y^2} = \frac{21}{10^3}$ (10)

10 a) $5\frac{1}{2} = 4\frac{y}{6}$ (9)

b) $(56{,}100 \div 187) + (15{,}718 \div 187) = y$ (384)

c) $58 \times 2 \times 273 = y$ (31,668)

d) $\frac{14}{10^2} \times \frac{3}{10^2} = y$ $(\frac{42}{10^4})$

e) $5\frac{1}{2} - 4\frac{7}{10} = y$ $(\frac{4}{5})$

f) $10 \times 7.936 = y$ (79.36)

g) $875 + 374 + 923 = y$ (2172)

h) $6\frac{3}{4} + 9\frac{1}{2} = y$ $(16\frac{1}{4})$

i) $3^4 \div 9 = y$ (3^2 or 9)

11 a) $\frac{7}{8} + \frac{3}{4} = y$ $(1\frac{5}{8})$

b) $16.58 + 8.28 + 787.54 + .56 = y$ (812.96)

c) $10 \times 29.2 = y$ (292)

d) $1\frac{5}{6} - 1\frac{1}{4} = y$ $(\frac{7}{12})$

e) $\frac{5}{9} \times \frac{3}{5} = y$ $(\frac{1}{3})$

f) $3 \times 3 \times 2{,}875 = y$ (25,875)

g) $595{,}161 \div y = 603$ (987)

h) $\frac{y}{12} = \frac{1}{4}$ (3)

i) $32 \times 2^2 = y$ (2^7 or 128)

Braintwisters

1. What four consecutive odd numbers, when added together, will equal 80? (17, 19, 21, 23)
2. Can you find any two prime numbers less than 100 whose sum is an odd number? (2 and 3 are the only pair.)

Review

SET I

Part A

1. Express in exponent form.

a) $5^4 \times 5^2 = \underline{5^6}$ e) $16 \div 8 = \underline{2^4 \div 2^3 = 2^1}$

b) $16 \times 2 = \underline{2^4 \times 2^1 = 2^5}$ f) $100 \times 1{,}000 = \underline{10^2 \times 10^3 = 10^5}$

c) $7^3 \div 7^2 = \underline{7^1}$ g) $4 \times 32 = \underline{2^2 \times 2^5 = 2^7}$

d) $125 \div 5^2 = \underline{5^3 \div 5^2 = 5^1}$ h) $81 \div 3 = \underline{3^4 \div 3^1 = 3^3}$

2. Find the number that n represents. The first one is done for you.

a) $\frac{1}{2} = \frac{n}{20}$ $n = 10$ d) $\frac{n}{5} = \frac{16}{40}$ $n = 2$

b) $\frac{1}{3} = \frac{n}{36}$ $n = 12$ e) $\frac{12}{18} = \frac{n}{6}$ $n = 4$

c) $\frac{3}{n} = \frac{30}{60}$ $n = 6$ f) $\frac{3}{5} = \frac{18}{n}$ $n = 30$

3. Arrange according to value, putting the smallest first. Example a is done for you.

a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ $\underline{(\frac{1}{4}, \frac{1}{3}, \frac{1}{2})}$ d) $\frac{15}{18}, \frac{5}{9}, \frac{2}{3}$ $(\frac{5}{9}, \frac{2}{3}, \frac{15}{18})$

b) $\frac{1}{3}, \frac{1}{5}, \frac{1}{6}$ $(\frac{1}{6}, \frac{1}{5}, \frac{1}{3})$ e) $\frac{5}{30}, \frac{2}{15}, \frac{1}{10}$ $(\frac{1}{10}, \frac{2}{15}, \frac{5}{30})$

c) $\frac{2}{4}, \frac{2}{5}, \frac{2}{3}$ $(\frac{2}{5}, \frac{2}{4}, \frac{2}{3})$ f) $\frac{3}{10}, \frac{1}{3}, \frac{7}{30}$ $(\frac{7}{30}, \frac{3}{10}, \frac{1}{3})$

4. What exponent does n stand for in $3^n = 9$? You know that $3 \times 3 = 9$. Then $3^2 = 9$ and $n = 2$. Copy the problems and find n for each.

a) $2^n = 8$, n = 3

b) $6^n = 216$, n = 3

c) $3^n = 243$, n = 5

d) $4^n = 1{,}024$, n = 5

e) $7^n = 343$, n = 3

f) $5^n = 625$, n = 4

g) $3^n = 81$, n = 4

h) $4^n = 64$, n = 3

i) $6^n = 1{,}296$, n = 4

j) $10^n = 100{,}000$, n = 5

5. Place parentheses in the following to make the sentences true. Example: $3 + 13 \div 4 = 4$, $(3 + 13) \div 4 = 4$

a) $3 \div 1 \times 3 = 9$
$(3 \div 1) \times 3 = 9$

b) $14 - 2 + 5 = 7$
$14 - (2 + 5) = 7$

c) $4 \times 5 + 2 = 28$
$4 \times (5 + 2) = 28$

d) $17 + 3 \times 4 = 80$
$(17 + 3) \times 4 = 80$

e) $6 - 2\frac{2}{3} + 1\frac{1}{3} = 4\frac{2}{3}$
$(6 - 2\frac{2}{3}) + 1\frac{1}{3} = 4\frac{2}{3}$

f) $\frac{3}{4} \times 5\frac{5}{6} - 2\frac{1}{8} = 2\frac{1}{4}$
$(\frac{3}{4} \times 5\frac{5}{6}) - 2\frac{1}{8} = 2\frac{1}{4}$

g) $3 + 3\frac{1}{2} - 1\frac{7}{10} = 4.8$
$3 + (3\frac{1}{2} - 1\frac{7}{10}) = 4.8$

h) $13 - 9\frac{3}{4} + 2\frac{5}{6} = \frac{5}{12}$
$13 - (9\frac{3}{4} + 2\frac{5}{6}) = \frac{5}{12}$

6. Estimate the two whole numbers the sum or product must be between. An example is done for you.

Example: $2\frac{1}{2} + 3\frac{1}{6}$, 5 and 7

a) $6\frac{1}{2} + 9\frac{7}{10}$ 15 and 17

b) 3.25×2.4 6 and 12

c) $\frac{5}{6} \times 4\frac{1}{2}$ 0 and 5

d) $5\frac{1}{2} + 7\frac{2}{5}$ 12 and 14

e) 8.9×7.50 56 and 72

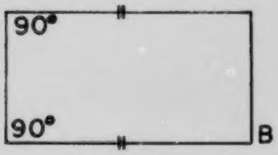
f) $12\frac{2}{3} + 15\frac{3}{6}$ 27 and 29

g) $23\frac{1}{7} \times \frac{4}{5}$ 0 and 24

h) $85\frac{3}{4} + 92\frac{5}{6}$ 177 and 179

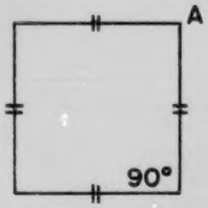
7. In each polygon below the sizes of some of the angles are shown. Write the measure of each angle whose vertex is named by a letter. Tell what each polygon would be called. Sides marked || are congruent. Example a is done for you.

a)



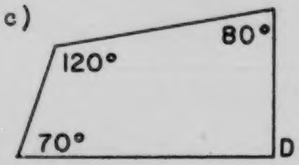
$m\angle B = 90$
Rectangle

b)



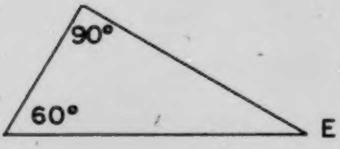
$m\angle A = 90$
Square

c)



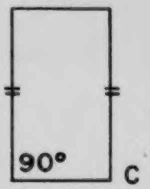
$m\angle D = 90$
Quadrilateral

d)



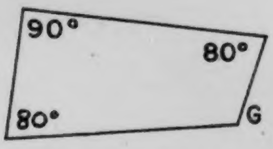
$m\angle E = 30$
Triangle (Scalene)

e)



$m\angle C = 90$
Rectangle

f)



$m\angle G = 110$
Quadrilateral

8. Complete the chart below.

Name of Star	Approximate distance in miles, using exponent form	Approximate distance in miles, without using exponent form
Procyon	65×10^2	65,000,000,000,000
Betelgeuse	18×10^{14}	1,800,000,000,000,000
Regulus	41×10^{13}	410,000,000,000,000
Altair	11×10^{13}	110,000,000,000,000
Vega	16×10^{13}	160,000,000,000,000
Alpha Centauri	$2^2 \times 3^3 \times 10^{10}$	1,080,000,000,000
Rigel	32×10^{14}	3,200,000,000,000,000

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. If Mrs. Jones cut $\frac{1}{2}$ of a pie into three equal pieces, each piece is what part of the whole pie? $\frac{1}{2} \times \frac{1}{3} = a$, $a = \frac{1}{6}$
   Each piece of pie is $\frac{1}{6}$ of the whole pie.
2. Jim has a triangle design on his garage door. The triangle has one side 8 inches long and the other two sides each 6 inches long. The angle opposite the 8 inch side is 80°. What is the sum of the measures of the other two angles? $80 + n = 180$ or $180 - 80 = n$, $n = 100$
   The sum of the measures of the other angles is 100.
3. Ruth lives $\frac{5}{8}$ mile from the library. Linda lives twice as far from the library as Ruth does. How far does Linda live from the library? $\frac{5}{8} \times 2 = d$, $d = \frac{10}{8}$ or $1\frac{2}{8}$ or $1\frac{1}{4}$
   Linda lives $1\frac{1}{4}$ miles from the library.
4. The length of the playground at the Pine Grove School is 150 ft. The width of this playground is 60 ft. The playground at the Recreation Center has the same area as the playground at the school. The width of the Center's playground is 90 ft. What must be the length of the playground at the Center? $60 \times 150 = n$, $n \div 90 = y$
   $y = 100$, $60 \times 150 = y \times 90$ The playground must have the length of 100 ft.
5. Mrs. Brown bought 14 yards of cloth to make curtains. She used $\frac{3}{4}$ of the material for the kitchen and bedrooms. How much material is left? $\frac{3}{4} \times 14 = r$, $r = 10\frac{1}{2}$, $14 - 10\frac{1}{2} = n$
   $n = 3\frac{1}{2}$. Mrs. Brown has $3\frac{1}{2}$ yards of material left.

6. A recipe for dessert for six persons calls for $\frac{3}{4}$ cup of sugar. How much sugar will be needed to make the dessert for three persons? $\frac{1}{2} \times \frac{3}{4} = r$, $r = \frac{3}{8}$
$\frac{3}{8}$ cup of sugar will be needed.

7. In 1950 the population of India was estimated to be $5 \times 2^3 \times 10^7$. Express this as a base ten numeral.
$5 \times 8 \times 10{,}000{,}000 = p$, $p = 400{,}000{,}000$ The population would be 400,000,000 people.

8. A planet moves around the sun. When it is closest, it is 70 million miles from the sun. When it is farthest, it is 90 million miles from the sun. What is its average distance from the sun? Write the average in exponent form.
$(70{,}000{,}000 + 90{,}000{,}000) \quad 2 = 80{,}000{,}000$ Its average distance is 80 million miles or 8×10^7.

9. Jack's sister in high school has an average of $2\frac{1}{3}$ hours of home work each school night. How many hours will she spend on home work each school week? $5 \times 2\frac{1}{3} = n$ $n = 11\frac{2}{3}$
Jack's sister spends $11\frac{2}{3}$ hours each week on home work.

Braintwisters

"Clock Arithmetic"

Is 10 + 6 always equal to 16? (No)

If it is 10 a.m. now, what time will it be 6 hours later? (4:00 p.m.)

In counting hours what happens on an ordinary clock after the hour hand gets to 12 ? (It goes to 1)

The answers to problems in Example A are correct only for "clock arithmetic".

a (1) $3 + 4 \quad 7$ (2) $7 + 9 = 4$ (3) $8 + 6 = 2$ (4) $11 + 7 = 6$

Use "clock arithmetic" to write the sum for these.

b (1) $9 + 6 = 3$ (2) $10 + 7 = 5$ (3) $11 + 11 = 10$ (4) $5 + 4 = 9$

Can you find products in "clock arithmetic"? (Yes)

Review

SET II

Part A

1. Tell which property is illustrated by each of these mathematical sentences. Write the first letter of each word that names the property. For example, write A P M for associative property of multiplication.

a) $(a + b) + c = a + (b + c)$ A P A

b) $c + d = d + c$ C P A

c) $a \times (d \times c) + (a \times d) \times c$ A P M

d) $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$ C P M

e) $(a + b) \times c = (a \times c) + (b \times c)$ D P M

f) $a \times b = a \times b$ None

g) $(b + c) \div a = (b \div a) + (c \div a)$ D P D

h) $a \times c = c \times a$ C P M

i) $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} + \frac{c}{d}$ None

j) $(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$ A P M

2. Write the repeated factor form and the numeral for each of the following. Example a is done for you.

a) $2^4 = 2 \times 2 \times 2 \times 2 = 16$

b) $4^3 = 4 \times 4 \times 4 = 64$

c) $6^2 = 6 \times 6 = 36$

d) $4^5 = 4 \times 4 \times 4 \times 4 \times 4 = 1024$

e) $10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1,000,000$

f) $8^3 = 8 \times 8 \times 8 = 512$

g) $5^4 = 5 \times 5 \times 5 \times 5 = 625$

3. Answer yes or no to each of the following questions.

a) Does $0.15 + 2.3 + 7 = \frac{15}{100} + \frac{23}{10} + \frac{70}{10}$? Yes

b) Does $4.2 \times 3.5 = \frac{42 \times 35}{10^2}$? Yes

c) Does $\frac{5 \times 6}{10^3} = 0.56$? No

d) Does $0.032 = \frac{4 \times 8}{1{,}000}$? Yes

e) Does $4 = \{\frac{4}{1}, \frac{8}{2}, \frac{12}{3}, \frac{16}{4}\}$ Yes

f) Does $\frac{8 + 8}{4 + 2} = 8$? No

g) Does $\frac{21}{15} = \frac{3 \times 7}{5 \times 3} = \frac{3}{5} \times \frac{7}{3}$? Yes

h) Does $7 \times \frac{2}{3} \times \frac{3}{6} = \frac{1}{7} \times \frac{2}{3} \times \frac{3}{6}$? No

i) Does $3\frac{1}{2} \times 5\frac{7}{8} = 20\frac{9}{16} + \frac{2}{16}$? No

j) Does $\frac{3}{3} \times \frac{4}{5} = \frac{4}{5} \times 1$? Yes

4. Write the following as decimals, fractions, and fractions with denominator in exponent form. Example a is done for you.

a) Two and four tenths $(2.4, \frac{24}{10}, \frac{24}{10^1})$

b) Twenty-four hundredths $(0.24, \frac{24}{100}, \frac{24}{10^2})$

c) Three and six hundredths $(3.06, \frac{306}{100}, \frac{306}{10^2})$

d) Four and thirty hundredths $(4.30, \frac{430}{100}, \frac{430}{10^2})$

e) Five and sixteen thousandths $(5.016, \frac{5016}{1000}, \frac{5016}{10^3})$

f) Three hundred twenty-four hundredths $(3.24, \frac{324}{100}, \frac{324}{10^2})$

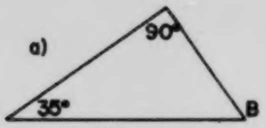
g) One hundred sixty-four tenths $(16.4, \frac{164}{10}, \frac{164}{10^1})$

h) Thirty-nine and seventy hundredths $(39.70, \frac{3970}{100},$ 39

i) Two thousand four and one tenth $(2004.1, \frac{20041}{10},$ 200

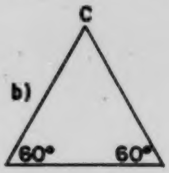
5. In each triangle below the sizes of some of the angles are shown. Write the measure of each angle whose vertex is named by a letter. Tell what each triangle would be called. Sides marked || are congruent. Example a is done for you.

a)



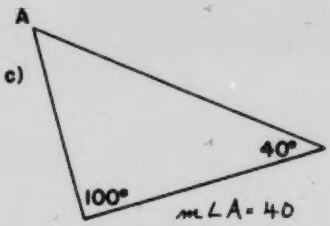
$m\angle B = 55$
Scalene

b)



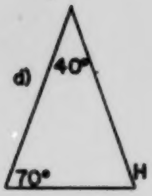
$m\angle C = 60$
Equilateral

c)



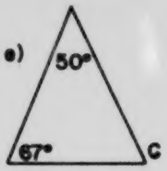
$m\angle A = 40$
Isosceles

d)



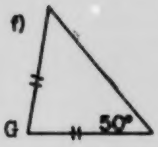
$m\angle H = 70$
Isosceles

e)



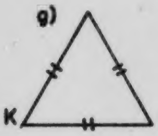
$m\angle C = 63$
Scalene

f)



$m\angle G = 80$
Isosceles

g)



$m\angle K = 60$
Equilateral

6. Some countries use the metric system to measure length. It is a decimal system as shown below.

10 millimeters (mm) = 1 centimeter (cm)
10 centimeters = 1 decimeter (dm)
10 decimeters = 1 meter (m)
1000 meters = 1 kilometer (km)

Fill in the blanks.

a) 1 mm = .1 cm
b) 1 mm = .01 dm
c) 1 mm = .001 m
d) 1 m = 1000 mm
e) 1 m = 100 cm

7. An astronomer was trying to find how many miles it is from the center of the earth to the center of the moon. This was his mathematical sentence, $(6 \times 10^1) \times (4 \times 10^3) = n$. How many miles is it? (240,000 miles)

8. The mass (not weight) of the earth is 6,000 million million million tons. Using exponent form this could be written as 6×10^n. What number does n represent? ($n = 21$)

9. Some scientists say the earth is about five billion years old. Write the age of the earth four different ways.
(5,000,000,000 = (5 × 1,000,000,000) = 5 × 10 × 10 × 10 × 10 × 10 × 10 × 10 × 10 × 10 = 5×10^9)

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. George bent a piece of wire to form a triangle. He found the size of one angle to be 90° and the size of another angle to be 30°. What will be the size of the third angle?
$180 - (90 + 30) = n$, $n = 60$, $90 + 30 = 120$, $180 - 120 = n$
$n = 60$ The size of the third angle will be 60°.

2. James can run one-fourth mile in $2\frac{1}{4}$ minutes. At this speed how long will it take to run one mile? $4 \times 2\frac{1}{4} = n$, $n = 9$
James can run one mile in 9 minutes.

3. Eddie rode his bicycle 3.7 miles in fifteen minutes. At this speed how far does Eddie travel in one hour? $4 \times 3.7 = r$
$r = 14.8$ Eddie travels 14.8 miles in one hour.

4. An airplane is traveling at $2^4 \times 5^2$ miles an hour. Express this as a base ten numeral. $16 \times 25 = s$ $s = 400$
The plane is traveling at 400 miles an hour.

5. In many countries a standard unit of measure is the meter A meter equals 1.0936 yards. How many yards more than 100 yards is a 100 meter race? $(1.0936 \times 100) - 100 = y$
$y = 9.36$, $1.0936 \times 100 = n$, $n - 100 = y$ $y = 9.36$
A 100 meter race is 9.36 yards longer than a 100 yard rac

6. Dan bought $\frac{1}{2}$ pound of lunch meat. The first day he ate $\frac{1}{4}$ of the lunch meat. How much lunch meat did he eat?
$\frac{1}{2} \times \frac{1}{4} = s$, $s = \frac{1}{8}$ Dan ate $\frac{1}{8}$ pound of lunch meat.

7. A musical tone is made by vibrations. When struck, string E vibrates 256 times a second. String C vibrates $1\frac{1}{2}$ ti as much as string E. How many times a second will string C vibrate? $256 \times 1\frac{1}{2} = n$, $\frac{256}{1} \times \frac{3}{2} = n$, $n = \frac{768}{2}$ or 384
String C vibrates 384 times per second.

Group Activities

Relays Using Sequence-- The object of the game is to discover the rule and to write the terms of a sequence. The teacher dictates the first three numbers of the sequence to the first player of each team. The first player writes them and then returns to his seat. After a brief time, during which the class discovers the rule of the sequence, the signal to begin is given. Each player, going up in turn, adds a term to the sequence. The relay continues until the teacher calls time. The winning team is the one with the longest correct sequence.

This game can be adapted for many kinds of interesting practice.

Review

SET III

Part A

1. Arrange in the order of size from least to greatest.

   a) 0.74, 0.014 , 1.40 , 0.7 (0.014, 0.7, 0.74, 1.40)

   b) 0.65 , 0.8 , 0.07 , 0.10 (0.07, 0.10, 0.65, 0.8)

   c) 1.1 , 1.32 , 1.0, 1.85 (1.0, 1.1, 1.32, 1.85)

2. Write the pair of equal fractions in each of the following. The first one is worked for you.

   a) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{8}$ ($\frac{1}{2}$, $\frac{4}{8}$)

   b) $\frac{3}{5}$, $\frac{9}{15}$, $\frac{3}{10}$ ($\frac{3}{5}$, $\frac{9}{15}$)

   c) $\frac{2}{3}$, $\frac{6}{4}$, $\frac{4}{6}$ ($\frac{2}{3}$, $\frac{4}{6}$)

   d) $\frac{5}{9}$, $\frac{15}{36}$, $\frac{20}{36}$ ($\frac{5}{9}$, $\frac{20}{36}$)

   e) $\frac{11}{4}$, $\frac{21}{8}$, $\frac{33}{12}$ ($\frac{11}{4}$, $\frac{33}{12}$)

   f) $\frac{5}{4}$, $\frac{10}{12}$, $\frac{15}{12}$ ($\frac{5}{4}$, $\frac{15}{12}$)

3. Find the number that the letter represents. Express answers in simplest form.

   a) $\frac{3}{4} + \frac{1}{8} = n$ $n = \frac{7}{8}$

   b) $\frac{5}{8} \times \frac{1}{3} = p$ $p = \frac{5}{24}$

   c) $3.6 \times 0.6 = r$ $r = 2.16$

   d) $\frac{4}{3} - \frac{4}{6} = d$ $d = \frac{2}{3}$

   e) $2.81 \times 0.7 = p$ $p = 1.967$

   f) $\frac{13}{16} - n = \frac{3}{4}$ $n = \frac{1}{16}$

   g) $16.3 \times 0.09 = b$ $b = 1.467$

   h) $6 \times 5\frac{1}{4} = c$ $c = 31\frac{1}{2}$

4. Find the number that n represents. Tell whether it is a prime or a composite number.

   Examples: $2^3 + 1 = n$, $n = 9$ composite; $2^3 - 1 = n$, $n = 7$ prime

   a) $9^2 - 1 = n$ $n = 80$ composite

   b) $2^5 - 1 = n$ $n = 31$ prime

   c) $3^3 + 1 = n$ $n = 28$ composite

   d) $4^2 + 1 = n$ $n = 17$ prime

   e) $6^2 + 1 = n$ $n = 37$ prime

   f) $5^2 - 1 = n$ $n = 24$ composite

   g) $7^3 + 1 = n$ $n = 344$ composite

   h) $4^5 - 1 = n$ $n = 1023$ composite

5. Express each of these numbers as a product of primes.

Example: $16 = 2 \times 2 \times 2 \times 2$

a) $125 = 5 \times 5 \times 5$

b) $81 = 3 \times 3 \times 3 \times 3$

c) $27 = 3 \times 3 \times 3$

d) $21 = 3 \times 7$

e) $12 = 2 \times 2 \times 3$

f) $68 = 2 \times 2 \times 17$

g) $39 = 3 \times 13$

h) $32 = 2 \times 2 \times 2 \times 2 \times 2$

i) $48 = 2 \times 2 \times 2 \times 2 \times 3$

j) $243 = 3 \times 3 \times 3 \times 3 \times 3$

Which of the above numbers could be written in exponent form as a power of two? Example: $16 = 2^4$. Write them. ($32 = 2^5$)
Which could be written in exponent form as a power of three? Write them. ($81 = 3^4$, $243 = 3^5$, $27 = 3^3$)

6. A ten foot ladder is leaning against a wall. The top of the ladder is 8 feet from the ground. The bottom of the ladder is 6 feet from the wall. What kind of triangle does this suggest? (Scalene) Where will the largest angle of that triangle be located: where the ladder touches the wall or ground, or where the ground and wall touch? (Where the ground and wall touch)

Part B

Write a mathematical sentence (or two sentences if necessary) and solve. Write an answer sentence.

1. Jill is cutting out a pattern. One piece has the shape of an equilateral triangle. One side of the triangle is 34 inches long. How many inches will Jill cut when she cuts out the triangle? $34 \times 3 = n$, $34 + 34 + 34 = n$, $n = 102$
Jill will cut 102 inches.

2. There were 36 children in Miss Heyler's class in January. Three-fourths of the class had perfect attendance. How many children had perfect attendance? $\frac{3}{4} \times 36 = a$, $a = 27$
27 children had perfect attendance.

3. A snail crawls $\frac{3}{8}$ of an inch in one minute and $\frac{5}{3}$ of an inch the next minute. How far does he crawl in the two minutes? $\frac{3}{8} + \frac{5}{3} = n$, $\frac{9}{24} + \frac{40}{24} = n$, $n = \frac{49}{24}$ or $2\frac{1}{24}$ The snail crawls $2\frac{1}{24}$ inches in two minutes.

4. Dan caught a lake trout that weighs $2\frac{1}{2}$ pounds. How many ounces does the trout weigh? $16 \times 2\frac{1}{2} = w$, $16 \times \frac{5}{2} = w$,
$w = \frac{80}{2}$ or $w = 40$ The trout weighs 40 ounces.

5. An astronaut is traveling around the earth at $5^2 \times 10^3$ miles an hour. He travels 75,000 miles each orbit. How long will it take him to make one orbit?
$25 \times 1{,}000 \times y = 75{,}000$ or $75{,}000 \div (25 \times 1{,}000) = y$ $y = 3$
One orbit will take 3 hours.

6. David's father weighs 196.5 pounds. David's weight is $\frac{1}{3}$ of his father's weight. What is David's weight? $w = 65.5$
$196.5 \times \frac{1}{3} = w$ David's weight is 65.5 pounds.

7. A dress pattern calls for $3\frac{3}{4}$ yards of material. Mother wishes to make dresses for Mary, Jan, and Denise. How much material does she need? $3 \times 3\frac{3}{4} = m$ $m = 11\frac{1}{4}$,
$3 \times \frac{15}{4} = m$ $m = \frac{45}{4}$ Mother needs $11\frac{1}{4}$ yards of material.

Braintwister

1. The following statements are true. What number base other than ten is used in each?
   a) I am $13_{\underline{?}}$ years old. In three years I will be a teen ager. (Base seven)
   b) There are $100_{\underline{?}}$ inches in one yard. (Base six)
   c) My birthday is in October, the $12_{\underline{?}}$ month of the year. (Base eight)

Chapter 4

INTRODUCING THE INTEGERS

PURPOSE OF UNIT

The purpose of this unit is to extend the concept of number. The concept which the children now have is based on experiences with three kinds of number: counting numbers, whole numbers, and fractional numbers. In this unit a fourth useful kind of number, the integers, are introduced.

The specific purposes are:

1. To introduce the meaning of integers
2. To introduce a symbolism for integers
3. To introduce the operations of addition and subtraction with integers by use of diagrams
4. To guide children in the use of integers and the realization of their importance

This unit is intended to provide an introduction to integers. The purpose is one of meaning and awareness rather than the development of skill. It is recommended that the teacher not drill for mastery of addition and subtraction of integers. This skill will be developed in later units.

MATHEMATICAL BACKGROUND

Introduction. The integers are a new kind of number to the children. The kinds of numbers which they have studied are:

counting numbers: $1, 2, 3, \ldots$
the whole numbers: $0, 1, 2, 3, \ldots$
fractional numbers: $\ldots, \frac{1}{2}, \ldots, \frac{8}{8}, \ldots, \frac{13}{7}, \ldots$

Each of these kinds of numbers arose historically out of a practical need. The integers arose from situations where counting or measuring in whole units was with respect to a fixed reference point from which the direction of counting or measuring was important. Examples of such situations are measuring temperature in degrees, or altitude in feet. One talks about a temperature of 30 degrees above zero or 30 degrees below zero, an altitude of 300 feet above sea level or 300 feet below sea level.

The integers will be denoted by

$$\ldots, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, \ldots$$

read: ..., "negative two," "negative one," "zero," "positive one," "positive two," ... . The superscripts " $^{-}$ " and " $^{+}$ " describe the relation of the integer to the reference number 0. Zero is written without a superscript. In the Pupils' Book no distinction will be made between integer as a number and integer as a symbol.

Pairs of integers such as ${}^{+}1$ and ${}^{-}1$, or ${}^{+}2$ and ${}^{-}2$ are said to be "opposites." Thus ${}^{-}2$ is the opposite of ${}^{+}2$, and ${}^{+}2$ is the opposite of ${}^{-}2$. Zero is considered to be its own opposite.

Using integers, one could speak of a temperature of ${}^{+}32$ (positive 32) degrees which we agree to mean 32 degrees above zero or an altitude of ${}^{-}300$ (negative 300) feet, which we agree means 300 feet below sea level.

Geometric representation. The integers may be represented geometrically as follows. A line is drawn with a series of equally spaced dots on it as in Figure 1.

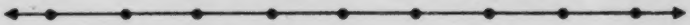


Figure 1

A dot is then selected somewhere near the middle of the line and labeled 0. The dots to the right of 0 are next labeled successively $^{+}1$, $^{+}2$, $^{+}3$, ... as far as convenient, and the dots to the left labeled similarly $^{-}1$, $^{-}2$, $^{-}3$, ... The result of this procedure is shown in Figure 2.

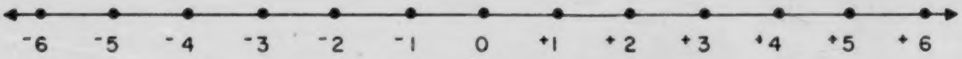


Figure 2

Figure 2 is a diagram of relations between integers. The equally spaced, labeled dots are the most important feature of the line. This is thought of not only as a number line, but also as a guide to the eye in moving from dot to dot. The equal spacing of the dots is important in this and similar diagrams later in the unit, because the addition and subtraction of integers will be presented graphically.

Figure 2 makes it apparent that the positive integers may be thought of as extending indefinitely to the right of 0, and the negative integers indefinitely to the left of 0. Pairs of opposite integers such as $^{-}4$ and $^{+}4$ are represented by pairs of dots spaced symmetrically to the right and left of 0.

The order of the integers. The integers shown in Figure 2 present themselves to the eye in a natural order from left to right. We say: $^{+}5$ is greater than $^{+}2$, because we must add $^{+}3$ to $^{+}2$ to get $^{+}5$. This is the same explanation we used with whole numbers to determine if one is larger than the other. That is, $a > b$ if some number c, greater than zero can be added to b so that $a = b + c$. $^{+}2$ is greater than $^{-}4$, because $^{+}6$ must be added to $^{-}4$ to get $^{+}2$. $^{-}3$ is greater than $^{-}10$, because $^{+}7$ must be added to $^{-}10$ to get $^{-}3$, or in mathematical symbols, $^{+}5 > {}^{+}2$, $^{+}2 > {}^{-}4$, $^{-}3 > {}^{-}10$. We may also say $^{+}2$ is less than $^{+}5$, $^{-}4$ is less than $^{+}2$, writing $^{+}2 < {}^{+}5$, $^{-}4 < {}^{+}2$. Note that in $^{+}5 > {}^{+}2$, $^{+}2 > {}^{-}4$, and $^{-}3 > {}^{-}10$, a *positive* integer was added to the smaller number to indicate that the other number was the larger of the two numbers. We have agreed that positive integers will be shown to the right of zero on the number line and negative integers to the left of zero, and that going from left to right is going in a positive direction, while going from right to left is going in a negative direction. We can say, then, that any integer represented by a dot on the number line is greater than any integer represented by a dot on the number line if this second dot lies to the left of the first dot.

Counting with integers. Starting with any integer, for example $^{+}2$, we can count forward by ones, $^{+}2, {}^{+}3, {}^{+}4, {}^{+}5, \ldots$ or backward by ones, $^{+}2, {}^{+}1, 0, {}^{-}1, {}^{-}2, \ldots$ as far as we please. We can also count forward or backward by other numbers, e.g., by threes: $^{+}2, {}^{+}5, {}^{+}8, {}^{+}11, \ldots$ or $^{+}2, {}^{-}1, {}^{-}4, {}^{-}7, \ldots$.

Addition of whole numbers can be described as repeated counting. Addition of integers may be similarly described. The only difference is that we may have to count backward (to the left) as well as forward (to the right). For example, the addition $^{+}5 + {}^{-}2 = {}^{+}3$ may be thought of as "Count forward five from 0 and then count backward two from positive five to positive three"; the addition $^{-}5 + {}^{+}2 = {}^{-}3$ may be thought of as "Count backward five from 0 and then count forward two from negative five to negative three." $^{-}5 + {}^{-}2$ would mean, "Count backward 5 from zero and then count backward two from negative 5 to negative 7." This method of approaching the operation of addition has some advantages, but it is not emphasized in this unit because it has the disadvantage that little knowledge of the nature of the operation of addition is imparted by it.

Representation of integers by arrows. An integer such as $^{+}5$ may be represented by drawing an arrow five spaces long parallel to the number line. In Figure 3, three such arrows are shown. We agree that arrows for positive numbers shall point to the right and arrows for negative numbers shall point to the left. The arrows all point to the right because $^{+}5$ is positive. Each arrow is placed so that it begins at one dot and ends at another dot on the number line. It may be started at any point that is convenient. An arrow is identified by writing the name of the integer which it represents directly above it as in Figure 3.

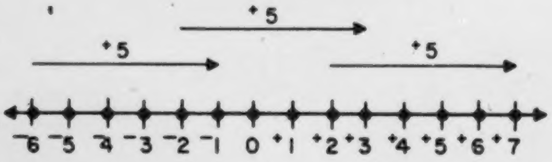


Figure 3

Figure 4 illustrates the arrows used in graphical representation of integers and the names used to describe the arrows.

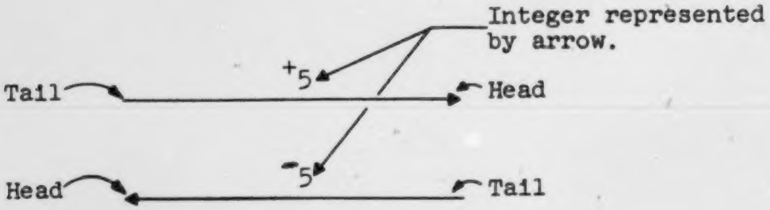


Figure 4

The number of spaces between the head and the tail of the arrow is called the measure of the arrow. In this unit it is a whole number. This conforms with what children already know about measurement; that is, that a measure is a whole number. The measure of the arrows in Figures 3 and 4 is 5 (a whole number).

In Figure 5, arrows representing $^{-}5$ and $^{-}2$ are shown. Note that they all point to the left. The measure of the $^{-}5$ arrow is 5. The measure of the $^{-}2$ arrow is 2.

In all graphical representations, positive is thought of as "to the right" and negative as "to the left."

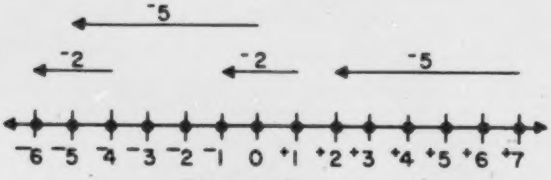


Figure 5

Renaming. Considerable use has been made previously of renaming a number. For example, the whole number three may be renamed $2 + 1$ or $4 - 1$ and the fraction $\frac{1}{2}$ may be renamed $\frac{1}{4} + \frac{1}{4}$ or $\frac{2}{3} - \frac{1}{6}$. In these names the symbols " + " and " - " denote the operations of addition and subtraction. Observe that $2 + 1$ and $4 - 1$ do not denote the performance of these operations on 2 and 1 or 4 and 1, but the result of performing the operations; "2 + 1" and " 4 - 1" are simply other names for the whole number 3. This fact is expressed by the mathematical sentences $2 + 1 = 3$ and $4 - 1 = 3$.

In this unit, the symbols " + " and " - " have other uses beside denoting operations on integers. When " + " is used as a superscript, as in " $^{+}2$," it is a part of the name of the number "positive two." It does not denote an addition. As a superscript, it serves to distinguish the integer positive two, $^{+}2$, from the whole number 2 and the fraction $\frac{2}{1}$. It also distinguishes positive two from its opposite, "negative two." Similar remarks apply to the symbol " - " when used as a superscript.

The operations of addition and subtraction of integers are denoted by the same symbols " + " and " - " as the like-named operations on whole numbers or fractions. The operations have very much the same properties. For example, addition of whole numbers and addition of fractions are both associative and commutative operations. Addition of integers is also an associative and commutative operation.

The associative property for addition of integers will not be developed in this unit, but extensive use will be made of the commutative property.

Renaming integers is conveniently done by using the arrow representation on the number line and renaming either from the figure or by counting. This is illustrated on pages 351, 352, and 355 of the Teachers' Commentary.

The mathematical sentences used to describe a renaming are similar to those used to describe renaming whole numbers or fractions. For example, the integer $^{+}5$ may be renamed $^{+}4 + {}^{+}1$ or $^{+}7 + {}^{-}2$ or $^{+}7 - {}^{+}2$. We shall see that $^{+}5$ also renames $^{+}7 - {}^{+}2$, $^{+}7 + {}^{-}2$ and $^{+}4 + {}^{+}1$. Observe that the expression $^{+}7 + {}^{-}2$ denotes the *result* of operating with addition on the integers $^{+}7$ and $^{-}2$. It does not denote the performance of the operation.

To *perform* the addition or to *add* $^{+}7$ and $^{-}2$ is to rename the integer $^{+}7 + {}^{-}2$ in as simple a way as possible. To indicate the performance of the operation, one writes $^{+}7 + {}^{-}2 = s$. Here s is called the *sum* of $^{+}7$ and $^{-}2$. It represents the *simplest name* for $^{+}7 + {}^{-}2$. When it is found that the sum is $^{+}5$, the addition has been performed. If the sum is unknown, it is called an unknown sum. The addition has been indicated, but not performed.

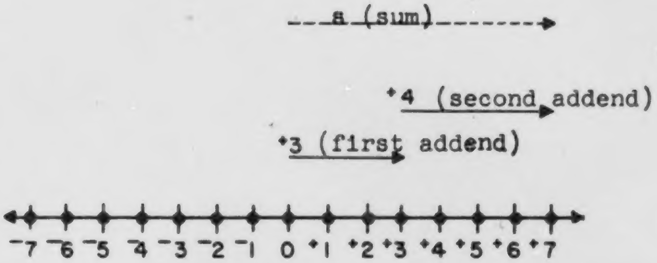
It is immaterial how the sum s is found. s may be found informally by counting back two from $^{+}7$ or s may be found graphically by the method presented in the following section. s could also be found by a suitable computation with the whole numbers 7 and 2 which the pupils will learn in later mathematics courses.

Whatever method is employed, the result is the same: the integer denoted by s in the mathematical sentence below is renamed $^{+}5$.

$$^{+}7 + {}^{-}2 = s$$

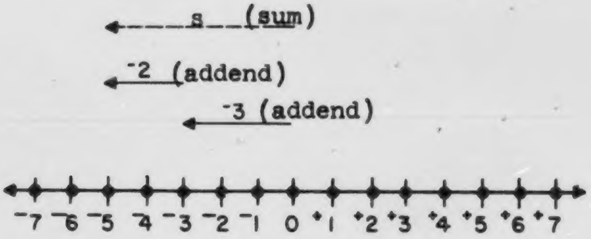
Graphical addition of integers. The procedure of graphical addition and subtraction using arrows and the number line diagram is chosen in this unit as best fitted to teach the meaning of the operations of addition and subtraction as applied to integers. The process of graphical addition is explained by the following diagrams.

Figures 6 through 9 illustrate a variety of simple addition sentences.


s (sum)
+4 (second addend)
+3 (first addend)
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 +1 +2 +3 +4 +5 +6 +7

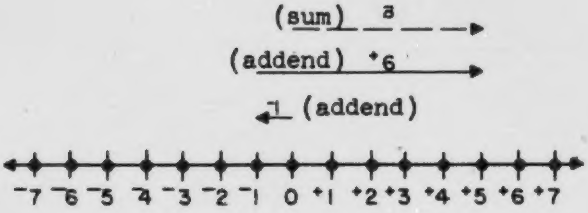
$^{+}3 + ^{+}4 = s$; $s = ^{+}7$

Figure 6


s (sum)
⁻2 (addend)
⁻3 (addend)
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 +1 +2 +3 +4 +5 +6 +7

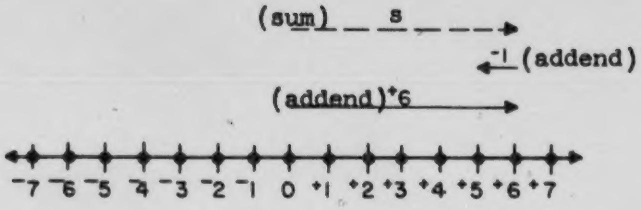
$^{-}3 + ^{-}2 = s$; $s = ^{-}5$

Figure 7


(sum) s
(addend) +6
⁻1 (addend)
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 +1 +2 +3 +4 +5 +6 +7

$^{-}1 + ^{+}6 = s$; $s = ^{+}5$

Figure 8



$^{+}6 + {}^{-}1 = s; \qquad s = {}^{+}5$

Figure 9

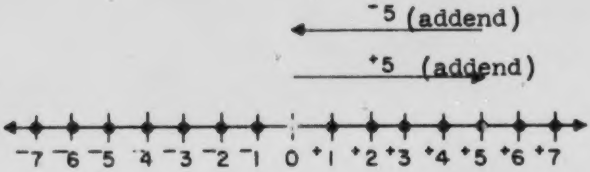
Certain general features of the process of graphical addition presented here should be noted.

(1) In an addition $b + c = s$ where b and c denote integers and s the unknown sum, the arrow for the first addend b is always drawn first. Its tail is directly above zero. It is drawn just above the number line. It is drawn to the right if b is a positive integer and to the left if b is a negative integer.

(2) The arrow for the second addend c is drawn next, just above the arrow that represents the first addend. Its tail starts directly above the head of the arrow for b and its direction depends on whether the integer c is positive or negative. If c is a positive integer, its direction is to the right. If c is a negative integer, its direction is to the left.

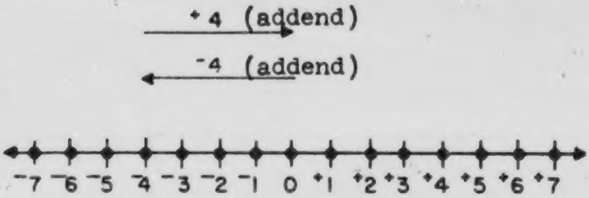
(3) The dotted arrow giving the unknown sum s is then drawn just above the arrow which represents the second addend. Its tail is always directly above the 0 dot. Its head is directly above the head of the second arrow. The sum s is the integer directly below its head.

Adding opposites. The fact that the sum of two opposite integers is always zero may be shown by graphical addition. Figures 10 and 11 indicate the procedure. No arrow for the sum can be drawn in these cases.



$^+5 + {}^-5 = s$; $s = 0$

Figure 10



$^-4 + {}^+4 = s$; $s = 0$

Figure 11

Subtraction of integers. Before the operation of subtraction in the set of integers is developed, the meaning and vocabulary of the operation of addition and subtraction of whole numbers should be reviewed. See Chapters 3 and 6 of Grade Four. No special language for subtraction of integers is needed. It is the same as the language for the subtraction of whole numbers as summarized in the following paragraphs.

In a mathematical sentence indicating addition as

(a) $5 + 13 = s$,

(1) the 5 and 13 are called "addends" or known addends; and

(2) the s is called the "unknown sum."

To perform the addition indicated by the sentence (a) is to find a more convenient name for $5 + 13$ thereby finding the sum s.

In an addition such as

(b) $5 + n = 18$

(1) the 5 and n are called "addends," 5 is the "known addend," and n is the "unknown addend";

(2) the 18 is the "sum."

Children have been taught that subtraction is the operation of finding an unknown addend. Sentence (b) may be rewritten as

(c) $n = 18 - 5$

and n then is found by subtracting.

The connection between sentence (b) and sentence (c) should be noted.

(1) The 18 is the sum.

(2) The 5 is the known addend, and the n is the unknown addend.

The sentences (b) and (c) have exactly the same mathematical meaning, and in both cases n is called an unknown addend.

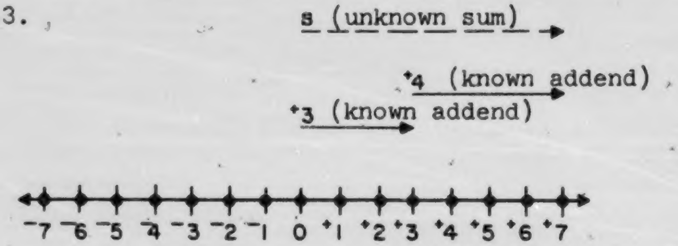
In the same way, if m denotes an integer, the six sentences below have the same mathematical meaning. In each sentence m denotes the unknown addend $^{+}2$.

(a) $^{+}3 + m = {}^{+}5$ (d) $^{+}5 = {}^{+}3 + m$

(b) $m + {}^{+}3 = {}^{+}5$ (e) $^{+}5 = m + {}^{+}3$

(c) $^{+}5 - {}^{+}3 = m$ (f) $m = {}^{+}5 - {}^{+}3$

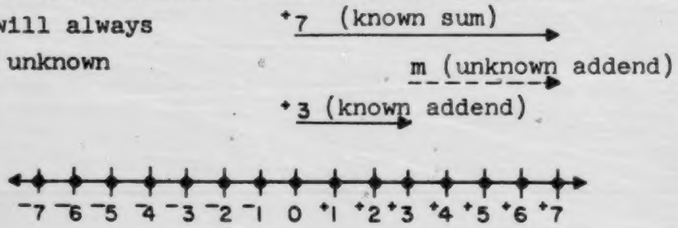
Graphical subtraction. Figure 12 is another graphical representation of an addition sentence. Figure 13 illustrates the naming of the unknown addend m in the sentences ${}^{+}7 = {}^{+}3 + m$ or ${}^{+}7 - {}^{+}3 = m$. Figure 12 is included here for purpose of comparison with Figure 13.


s (unknown sum)
+4 (known addend)
+3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

$${}^{+}3 + {}^{+}4 = s; \qquad s = {}^{+}7$$

Figure 12

Note the similarity of the location of the arrow for the sum, and the arrows for the first and second addends. Note also that in Figure 12 the unknown sum is sketched as a "dotted" arrow and in Figure 13 the unknown addend is sketched as a "dotted" arrow. A dotted arrow will always represent the unknown integer.


+7 (known sum)
m (unknown addend)
+3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

$${}^{+}7 = {}^{+}3 + m; \qquad m = {}^{+}4$$

Figure 13

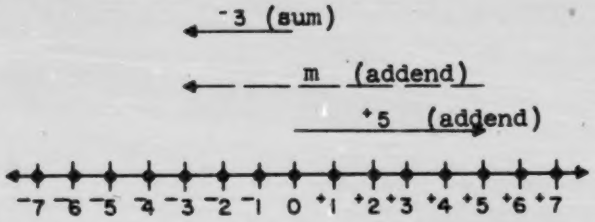
The procedure for drawing the arrow diagram of the sentence ${}^{+}7 = {}^{+}3 + m$ of Figure 13 is:

(1) Draw the ${}^{+}3$ arrow for the known addend just above the number line with its tail directly above 0.

(2) Draw the ${}^{+}7$ arrow for the sum at the top as in arrow diagrams for addition. This arrow has its tail directly above 0 and is drawn to the right because ${}^{+}7$ is a positive integer.

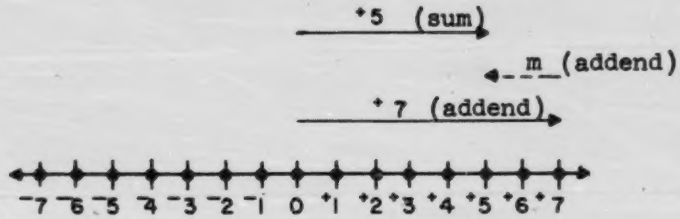
(3) Draw a dotted arrow from a point directly above the head of the $^{+}3$ arrow to a point directly below the head of the $^{+}7$ arrow. The "dotted" arrow represents the unknown addend. Thus $m = {}^{+}4$.

Figures 14 and 15 also illustrate the subtraction of integers. In each case the construction procedure follows the pattern used for the construction of Figure 13. Other arrow diagrams appear with their explanations on pages 373-376 of this Teachers' Commentary.



$^{-}3 = {}^{+}5 + m$; $^{-}3 - {}^{+}5 = m$; $m = {}^{-}8$

Figure 14



$^{+}5 = {}^{+}7 + m$; $^{+}5 - {}^{+}7 = m$; $m = {}^{-}2$

Figure 15

Subtraction of integers can be easily illustrated with arrow diagrams as was done above. Pupils are already familiar with the language of subtraction using whole numbers; that is, "In a subtraction problem, I am trying to find the unknown addend." Children should identify the addends and the sums in many mathematical sentences involving integers before drawing arrow pictures of them. It might be most helpful to have children rewrite subtraction sentences as addition sentences; e.g. $^{-}3 - {}^{+}6 = m$ or $^{-}3 = m + {}^{+}6$ or $m + {}^{+}6 = {}^{-}3$. This need not be done, of course, especially if children are very adept at recognizing the addends and the sum in a subtraction sentence which contains integers.

Relations between addition and subtraction. Subtraction of an integer is used as undoing the addition of that integer, just as subtraction of a whole number undoes the addition of that whole number. This is important because it shows that the operation of subtraction may be dispensed with in working with integers; every subtraction may be replaced by finding an unknown addend in an addition sentence. For example, the subtraction sentence $^{-}5 - {}^{+}7 = m$ is replaced by $^{-}5 = m + {}^{+}7$ in which $^{-}5$ is the sum, $^{+}7$ the known addend, and m the unknown addend.

Other symbols for positive integers. In later work in mathematics, when meaning has been developed, the symbols $^{+}1, {}^{+}2, {}^{+}3, \ldots$ which denote positive integers are usually written without superscripts thus: $1, 2, 3, \ldots$. This substitution of the symbol " 1 " for " $^{+}1$," " 2 " for " $^{+}2$," etc. does not mean that the whole number three and the positive integer $^{+}3$ are the same *number*. Later in his study of mathematics the pupil may find that the counting numbers and positive numbers are used interchangeably although they are different numbers. The following three paragraphs show why this may be done but it is not intended that this be presented to the pupils at this time.

Let us establish a one-to-one correspondence between the counting numbers and the positive integers. This is done by simply saying that 1 and $^{+}1$ correspond to each other, 2 and $^{+}2$ correspond to each other, etc. If the sets of counting numbers and positive integers are sets A and B respectively, the one-to-one correspondence may be indicated this way:

$$\begin{array}{rcccccccc}
A = \{ & 1, & 2, & 3, & 4, & \ldots, & a, & \ldots\} & \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\
B = \{ & ^{+}1, & ^{+}2, & ^{+}3, & ^{+}4, & \ldots, & ^{+}a, & \ldots\}. &
\end{array}$$

Now we consider sums and products of two numbers in set A and the corresponding numbers in set B. The sum of any two numbers in set A is a number in set A and the sum of any two numbers in set B is a number in set B. If we find the sum of any two numbers of set A and the sum of the two corresponding numbers of set B then the sums will be corresponding numbers in sets A and B. For example $4 \longleftrightarrow {}^{+}4$, $5 \longleftrightarrow {}^{+}5$, $4 + 5 = 9$, ${}^{+}4 + {}^{+}5 = {}^{+}9$ and 9 and ${}^{+}9$ are corresponding numbers in sets A and B respectively. Similarly if we find the product of any two numbers of set A and the product of the two corresponding numbers of set B, then the products will be corresponding numbers in sets A and B. For example $4 \times 9 = 36$ and ${}^{+}4 \times {}^{+}9 = {}^{+}36$ and 36 and ${}^{+}36$ are corresponding numbers in sets A and B respectively.

The preceding paragraph may be summarized by saying the sets A and B are isomorphic, or they are abstractly identical, and the numbers in the two sets may be used interchangeably. Consequently the positive integers may be used for all purposes for which the counting numbers are used. As numbers, they are different from the counting numbers, but they are operated on and ordered in precisely the same way.

If the pupils are to realize fully that the integers are new numbers (i.e., new in the sense that they are introduced after the counting numbers) it is necessary that the superscript " + " in the symbol for a positive integer be retained. If it is not retained the numbers 2 and ${}^{+}2$ will likely be identified with each other without any justification. Consequently it is recommended the superscript be used throughout this unit.

TEACHING THE UNIT

INTRODUCTION TO INTEGERS

Objective: To introduce the meaning of and symbolism for *integers*, a new kind of number

To introduce the arrow "→" or "←" as a means of diagraming an integer on a number line

Materials: Number lines made on paper with dots labeled with integers for teachers to fold to show oppositeness of pairs of integers

Dittoed copies of number lines for children (See suggestion on page 355)

Number lines made on cardboard to use on the chalk tray or bulletin board as a time-saver. These may be made in sections and fastened together with masking tape for ease in folding for storage

Vocabulary: Integer; " $^{+}5$ " read "positive five;" " $^{-}2$ " read "negative two"; arrow diagram; measure of arrow; direction of arrow; pair of opposites; head of arrow; tail of arrow; reference dot

Suggested Teaching Procedure:

> The method suggested for teaching is that of *discovery*.
>
> * * *
>
> The integers should be introduced in a class discussion, followed by having the children read the material written for them on pages 193 and 194 in their books and doing Exercise Set 1. This Exercise Set should then be discussed in class after which children may read page 197 in their books and complete Exercise Set 2 independently.

Present a line on which dots are located at equal intervals as shown in Figure 16. Arrows are used at the end of the line to show that it extends in both directions without ending.

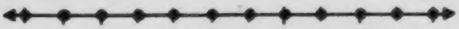


Figure 16

The dots on the line should then be numbered as the teacher says, "We are going to put some symbols for new numbers on this line." The zero should be located first, then pairs of dots so the children will understand the relationship of the numbers of a pair. Locate the pair $^{+}1$ and $^{-}1$ next, then $^{+}2$ and $^{-}2$, etc. After zero and several pairs are located, call them integers and read " $^{+}2$ " as "positive two," " $^{-}2$ " as "negative two," etc. See Figure 17.

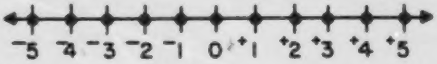


Figure 17

The similarity of this number line to other scales should be discussed. These scales and others should be drawn and the method of indicating direction from a reference dot made clear:

(a) Thermometer: Scale drawn with language "below zero" and "above zero" indicated with " + " and " - ."

(b) Scores: Some games result in scores that are "in the hole" or "out of the hole." Scores "in the hole" are represented with negative integers while scores "out of the hole" are represented with positive integers.

(c) Altitude above and below sea level. Altitudes above sea level are represented with positive integers and below sea level with negative integers.

It should be brought out by questions that all of these lines have dots indicated at equal intervals, have a definite reference dot, have dots marked in pairs on each side of the reference dot, and have pairs of dots marked to indicate oppositeness. It should also be clear that by agreement either direction may be positive if the opposite direction is negative.

Connect the pairs of number symbols as shown in Figure 18. Discuss the fact that integers appear in pairs and that the members of the pair number dots the same distance from zero. $^{-}1$ is paired with $^{+}1$, $^{-}3$ with $^{+}3$, and zero with itself.

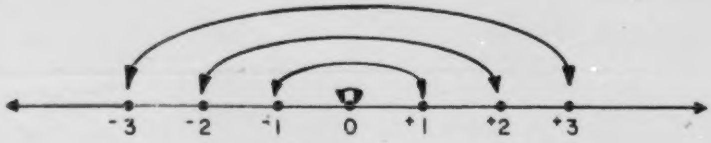


Figure 18

Emphasize the oppositeness of the members of each pair of integers. Show this oppositeness by folding a number line at zero so that $^{+}1$ falls on $^{-}1$, etc. Discuss symmetry of dots to the right and left of zero, that $^{-}3$ is the opposite of $^{+}3$, that $^{+}8$ is the opposite of $^{-}8$, etc. And, that zero is its own opposite.

Discuss with the pupils the order relations of the integers. You will recognize that it is not possible to define "greater than" and "less than"

> in terms of addition at this time since the pupils have not yet studied addition of integers. After addition has been studied you can show $a > b$ if there is some positive integer c so that $a = b + c$. For example $^{+}2 > {}^{-}3$ since $^{+}2 = {}^{-}3 + {}^{+}5$; $^{-}6 > {}^{-}10$ since $^{-}6 = {}^{-}10 + {}^{+}4$; $0 > {}^{-}3$ since $0 = {}^{-}3 + {}^{+}3$. But for the present, the meaning of "greater than" and "less than" for integers will need to be made clear intuitively by use of the number line for integers. The discussion with the pupils might go somewhat as suggested in the next paragraphs. In the presentation use the number line which shows the integers.

Is the whole number 5 greater than the whole number 2? (Yes. The pupils may say this is true because they must add to 2 in order to get 5 or they may say it is true because 5 is to the right of 2 on the number line.)

Does the greater one of any two whole numbers always label a point farther to the right than the other one of the whole numbers? (Yes.)

Are all the other whole numbers greater than the whole number 0? (Yes. $1 > 0$, $2 > 0$, etc.)

We want to be able to tell when one integer is greater than another integer. What would be an easy way to do this? (The integer that is farther to the right on the number line is greater than the other.)

We can write $^{+}2 > {}^{+}1$, $^{+}4 > {}^{+}3$, $^{+}6 > {}^{+}5$, and so on. Also we can write $^{+}1 < {}^{+}2$, $^{+}3 < {}^{+}4$, $^{+}5 < {}^{+}6$.

Now think of two of the negative integers, for example $^{-}3$ and $^{-}5$. Which one is farther to the right? ($^{-}3$) Then we will say that $^{-}3$ is the greater and write $^{-}3 > {}^{-}5$.

Which of the two integers $^{-}4$ and $^{-}1$ is the greater? ($^{-}1$ is greater than $^{-}4$ since $^{-}1$ is to the right of $^{-}4$.) We can write $^{-}1 > {}^{-}4$ or $^{-}4 < {}^{-}1$.

The order relations between two negative integers may be harder for the pupils to accept than the order relations between the positive integers. It may be helpful to lead to this by first talking to them about the order relations between the positive integers and negative integers. The thermometer scale and games which score "in the hole" and "out of the hole" may help. Try such questions as: (1) Is it warmer when the temperature is 2 degrees above zero than when it is 3 degrees below zero?, and (2) Is a score of "10 out of the hole" better than a score of "10 in the hole"?. Then from these order relations proceed to those between two negative integers.

Try such questions as: (1) Is it warmer when the temperature is 5 degrees below zero than when it is 9 degrees below zero?, (2) Is a score of "10 in the hole" better than a score of "20 in the hole"?.

The correct answers to these questions should help the pupils recognize that the greater of two negative numbers is represented by a dot farther to the right (or farther up on the thermometer scale if this is represented on a vertical line).

Pupils may use "count forward" or "count to the right" interchangeably. Also, "count backward" and "count to the left" will have the same meaning. The following discussion questions should assist in clarifying order relations of the integers.

Suggestions to stimulate discussion:

(1) Why is the " $^{-}$ " symbol needed as a part of " $^{-}5$ "? (To distinguish " $^{-}5$ " from " $^{+}5$ " and to locate a particular dot on our number line)

(2) Are the integers to the right of zero greater or less than zero? (Greater) Why?

(3) Are the integers to the right of " $^{+}7$ " greater or less than " $^{+}7$ "? (Greater)

(4) Are the integers to the left of zero greater or less than zero? (Less) To the left of " $^{-}3$ "? (Less)

(5) Are the integers to the right of any integer n greater than or less than n? (Greater)

(6) Are the integers to the left of any integer n greater or less than n? (Less)

(7) How many integers are to the right of zero? to the left of zero? (More than you can count)

(8) How many integers are to the left of positive four? to the right of positive five? (More than you can count)

(9) Count forward five spaces from " $^{+}3$ "; from " $^{-}2$ "; from " 0 "; etc. Count backward ten spaces from " $^{-}5$ "; from " $^{+}2$ "; etc.

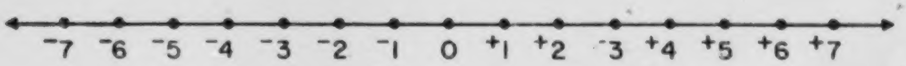
(10) Count forward from zero. Count backward from zero.

(11) Write the integers in set notation.
$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\}$

Chapter 4

INTRODUCING THE INTEGERS

A NEW KIND OF NUMBER



This is a number line. It has a point labeled 0 just like the number line you saw in Chapter 2. But the other points on this one have new labels. These labels are the names of some new numbers we are going to study.

Find the dot numbered 0. As your eye moves to the right, the dots are labeled $^{+}1$, $^{+}2$, $^{+}3$, ... As your eye moves to the left from zero, the dots are numbered $^{-}1$, $^{-}2$, $^{-}3$, ... The dots on the line represent numbers that are called integers. The set of integers is

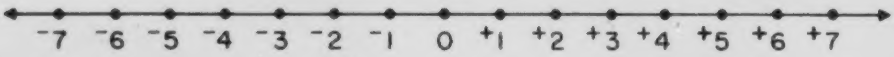
$$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\} .$$

The integer " $^{+}2$ " is read "positive two." The integer " $^{-}7$ " is read "negative seven." The 0 is read "zero." It is neither positive nor negative.

Integers on the line can be thought of in pairs. $^{+}6$ is paired with $^{-}6$; $^{-}1$ is paired with $^{+}1$; $^{+}10$ is paired with $^{-}10$; and zero is paired with itself.

These pairs are called opposites. $^{+}12$ is the opposite of $^{-}12$; $^{+}4$ is the opposite of $^{-}4$; $^{-}1$ is the opposite of $^{+}1$; and zero is its own opposite.

INTEGERS AND A NUMBER LINE



A part of a number line is shown above. It may be extended and numbered with integers as far as you wish. Try to imagine how many integers there are.

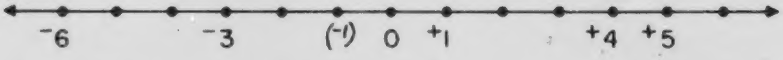
Look at the dot labeled $^{+}5$. Positive five is greater than $^{+}4$, or 0, or $^{-}3$, or $^{-}8$. In fact, positive five is greater than any integer which labels any dot to the left of the dot labeled by $^{+}5$. The mathematical sentences ... $^{+}5 > {}^{+}4$; $^{+}5 > {}^{+}3$; ... $^{+}5 > 0$; $^{+}5 > {}^{-}1$; ... $^{+}5 > {}^{-}4$; ... are ways of writing this.

Look again at the dot labeled $^{+}5$. Positive five is to the left of and less than $^{+}6$, or $^{+}7$, or $^{+}127$. In fact, positive five is less than any integer to its right. The mathematical sentences $^{+}5 < {}^{+}6$; $^{+}5 < {}^{+}7$; ... $^{+}5 < {}^{+}19$; ... $^{+}5 < {}^{+}127$; ... are ways of writing this fact.

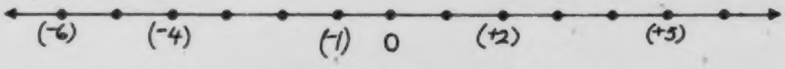
An integer represented by a dot on the number line is greater than any integer represented by a dot to its left; an integer represented by a dot is less than any integer represented by a dot to its right on the number line.

Exercise Set 1

1. Draw a number line like the one below and beneath it write the integers that are missing.



2. On a number line like the one below, locate and write each of these integers: $^{-}1$, $^{+}5$, $^{-}4$, $^{+}2$, $^{-}6$ below the dot which it labels.



3. Match each set with its best description:

a. $E = \{1, 2, 3, 4, \ldots\}$ (d) Whole numbers

b. $P = \{\ldots\ ^{-}3, ^{-}2, ^{-}1, 0, ^{+}1, \ldots\}$ (c) Fractional numbers

c. $N = \{\ldots \frac{1}{2}, \ldots \frac{7}{8}, \ldots \frac{9}{3}, \ldots\}$ (b) Integers

d. $T = \{0, 1, 2, 3, \ldots\}$ (a) Counting numbers

4. Copy and complete the following sentences by writing the correct symbol in the blank space, " >," " <," or " =."

a. $^{+}3$ (<) $^{+}5$

b. $^{-}12$ (<) $^{-}4$

c. $^{-}8$ (<) $^{+}6$

d. $^{+}1$ (>) $^{-}19$

e. $^{-}16$ (>) $^{-}32$

f. $^{+}479$ (>) $^{+}421$

g. $^{+}89$ (<) $^{+}95$

h. $^{-}26$ (=) $^{-}26$

i. $^{-}3$ (>) $^{-}5$

j. 0 (>) $^{-}7$

5. Name the integer that is

a. 3 greater than $^{-}12$. ($^{-}9$)

b. 7 less than 0. ($^{-}7$)

c. 4 greater than $^{+}16$. ($^{+}20$)

d. 5 less than $^{-}2$. ($^{-}7$)

e. 6 greater than 0. ($^{+}6$)

f. 2 less than $^{+}9$. ($^{+}7$)

6. Arrange the members of the following sets in the order they would appear on a number line from least to greatest:

$P = \{^{+}4, ^{-}6, 0\}$ $(\{^{-}6, 0, ^{+}4\})$

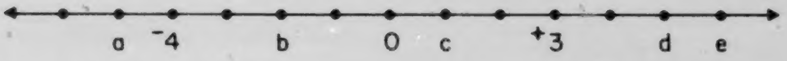
$F = \{^{-}19, ^{+}2, ^{+}17, ^{-}36\}$ $(\{^{-}36, ^{-}19, ^{+}2, ^{+}17\})$

$M = \{^{-}1, ^{+}5, ^{+}3, ^{-}7, ^{-}20\}$ $(\{^{-}20, ^{-}7, ^{-}1, ^{+}3, ^{+}5\})$

$R = \{^{+}13, ^{-}11, ^{+}1, ^{-}31, ^{-}3\}$ $(\{^{-}31, ^{-}11, ^{-}3, ^{+}1, ^{+}13\})$

$T = \{^{-}26, ^{+}4, ^{+}9, ^{+}12, ^{-}2\}$ $(\{^{-}26, ^{-}2, ^{+}4, ^{+}9, ^{+}12\})$

7. Compare the numbers a, $^{-}4$, b, 0, c, e, $^{+}3$, and d which are pictured on the number line below. Copy and fill in the mathematical sentences. Use the symbols ">," "<," or "=."



a. a $\underline{(<)}$ $^{-}4$

b. b $\underline{(<)}$ 0

c. e $\underline{(=)}$ e

d. e $\underline{(>)}$ $^{+}3$

e. d $\underline{(>)}$ b

f. $^{-}4$ $\underline{(>)}$ a

g. 0 $\underline{(>)}$ a

h. c $\underline{(>)}$ b

i. $^{+}3$ $\underline{(>)}$ $^{-}4$

8. **Name the integers which are**

a. greater than $^{-}2$ and less than $^{+}4$. $(^{-}1, 0, ^{+}1, ^{+}2, ^{+}3)$

b. less than $^{-}5$ and greater than $^{-}10$. $(^{-}6, ^{-}7, ^{-}8, ^{-}9)$

c. greater than $^{-}72$ and less than $^{-}67$. $(^{-}71, ^{-}70, ^{-}69, ^{-}68)$

d. less than $^{+}4$ and greater than $^{-}1$. $(^{+}3, ^{+}2, ^{+}1, 0)$

e. greater than $^{+}9$ and less than $^{+}10$. (none)

9. Describe these sets of integers as was done in exercise 8 a-e.

a. $\{^{-}7, ^{-}6, ^{-}5, ^{-}4\}$ (greater than $^{-}8$ and less than $^{-}3$)

b. $\{^{+}11, ^{+}12, ^{+}13\}$ (greater than $^{+}10$ and less than $^{+}14$)

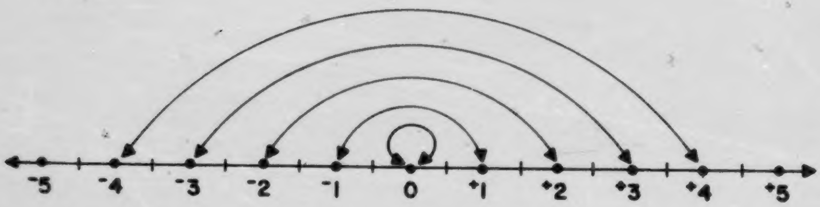
c. $\{^{+}1\}$ (greater than 0 and less than $^{+}2$)

d. $\{^{-}10\}$ (greater than $^{-}11$ and less than $^{-}9$)

INTEGERS AND THEIR OPPOSITES

Expressions like "five degrees below zero," "five hundred feet above sea level," "two points in the hole," and "three yards loss" are very common in our language. We may use integers to express these ideas:

Five degrees below zero	$^{-}5$ degrees
Five hundred feet above sea level	$^{+}500$ feet
Two points in the hole	$^{-}2$ points
Three yard loss	$^{-}3$ yards



Look at the diagram above. Each integer is one of a pair. $^{+}4$ and $^{-}4$ form a pair. Their dots are the same distance from the zero dot. $^{+}4$ is the opposite of $^{-}4$; $^{-}4$ is the opposite of $^{+}4$. Zero is its own opposite.

Every integer has an opposite.

Exercise Set 2

1. Which of the number pairs below are opposites: (a, e, f)

a. $^{-}3$; $^{+}3$ c. $^{+}7$; $^{-}2$ e. 0; 0
b. $^{-}5$; $^{-}5$ d. $^{-}8$; 0 f. $^{-}14$; $^{+}14$

2. Choose the greater integer, if possible, from each pair of opposites below.

a. $^{-}13$; $^{+}13$ ($^{+}13$) c. $^{-}7$; $^{+}7$ ($^{+}7$) e. $^{+}9$; $^{-}9$ ($^{+}9$)
b. $^{+}6$; $^{-}6$ ($^{+}6$) d. 0; 0 (no answer) f. $^{-}112$; $^{+}112$ ($^{+}112$)

3. Write in words the opposite meaning of:

a. 12° below zero
(12° above zero)
b. "5 in the hole"
("5 out of the hole")
c. \$10 profit
(\$10 loss)
d. 270 feet above sea level
(270 feet below sea level)
e. 15° north of the equator
(15° south of the equator)
f. 110° east of Prime Meridian
(110° west of Prime Meridian)

4. Write the opposite for each integer below.

a. $^{+}3$ ($^{-}3$) c. 0 (0) e. $^{+}77$ ($^{-}77$)
b. $^{-}7$ ($^{+}7$) d. $^{-}2$ ($^{+}2$) f. $^{-}256$ ($^{+}256$)

5. Use an integer to help describe each of the following.

a. Twenty degrees above zero ($^{+}20$) degrees
b. Two hundred feet below sea level ($^{-}200$) feet
c. Two points in the hole ($^{-}2$) points
d. Forty degrees south of the equator ($^{-}40$) degrees

6. Write an expression similar to those given in exercise 5 for each of these integers. (Answers will vary)

a. $^{+}17$ c. 0 e. $^{-}7$
b. $^{-}22$ d. $^{+}126$ f. $^{+}21$

7. Write numerals for:

a. negative two thousand twenty-two. ($^{-}2,022$)

b. positive five hundred fifty. ($^{+}550$)

c. negative sixty-one. ($^{-}61$)

d. positive eighty-nine. ($^{+}89$)

e. negative one thousand one. ($^{-}1,001$)

f. positive ten thousand four hundred ninety. ($^{+}10,490$)

8. Write words for:

a. $^{-}317$. (negative three hundred seventeen)

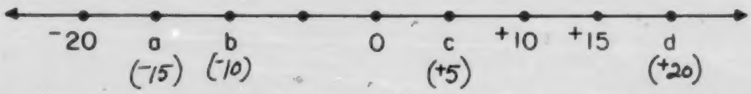
b. $^{+}203$. (positive two hundred three)

c. $^{-}6,060$. (negative six thousand sixty)

d. $^{+}910$. (positive nine hundred ten)

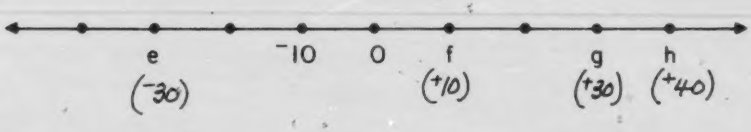
9. What integers are represented by the points labeled by letters on these number lines?

(a)



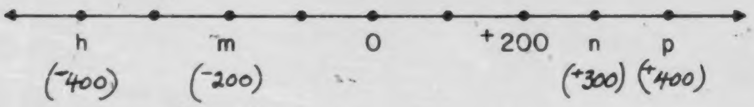
$^{-}20$
a
($^{-}15$)
b
($^{-}10$)
0
c
($^{+}5$)
$^{+}10$
$^{+}15$
d
($^{+}20$)

(b)



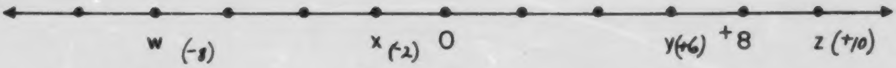
e
($^{-}30$)
$^{-}10$
0
f
($^{+}10$)
g
($^{+}30$)
h
($^{+}40$)

(c)



h
($^{-}400$)
m
($^{-}200$)
0
$^{+}200$
n
($^{+}300$)
p
($^{+}400$)

(d)



w ($^{-}8$)
x ($^{-}2$)
0
y ($^{+}6$)
$^{+}8$
z ($^{+}10$)

10. Write these sets of integers.

a. greater than $^{-}1$ and less than $^{+}1$ $(\{0\})$

b. less than $^{-}3$ and greater than $^{-}6$ $(\{^{-}5, ^{-}4\})$

c. greater than $^{+}5$ and less than $^{-}5$ $(\{\ \})$

d. less than 0 and greater than 0 $(\{\ \})$

11. Write these numerals.

a. negative three hundred fifty-four ($^{-}354$)

b. negative six thousand eight ($^{-}6,008$)

c. positive twenty three thousand ($^{+}23,000$)

d. positive forty seven thousand two hundred ($^{+}47,200$)

e. negative eight hundred four ($^{-}804$)

f. negative five thousand nine ($^{-}5,009$)

g. positive two thousand twenty ($^{+}2,020$)

h. negative twenty-five ($^{-}25$)

12. Write words for:

a. $^{+}29$ (positive twenty-nine)

b. $^{-}4,008$ (negative four thousand eight)

c. $^{-}8$ (negative eight)

d. $^{+}606$ (positive six hundred six)

e. $^{+}45$ (positive forty-five)

f. $^{-}370$ (negative three hundred seventy)

g. $^{+}8,001$ (positive eight thousand one)

h. $^{-}2,300$ (negative two thousand three hundred)

ARROW DIAGRAMS

Suggested Teaching Procedure:

Draw a number line with dots labeled with integers on the chalkboard or place one previously prepared of tagboard on the chalktray so that writing may be done above it. The latter suggestion is a real time-saver.

Have a child begin at $^{+}2$ on the number line, count 3 spaces to the right, and tell the number that labels the dot where he stops. Repeat, beginning at other labeled dots and counting to the right and to the left, e.g., "At what dot will you be if you start at $^{-}2$ and count to the left 3 spaces?"

The children should make arrows above the number line to record these counts. The arrows begin at the first dot counted and stop at the last dot counted. The heads of the arrows show the direction of the count. See Figure 19. Arrow a in Figure 19 shows the result of beginning at $^{+}2$ and counting 3 spaces to the right on the number line. Arrow b shows the result of beginning at $^{+}7$ and counting 6 spaces to the left.

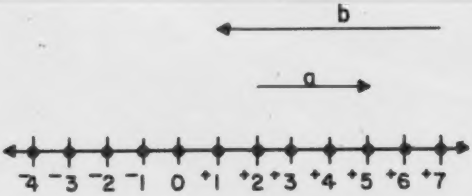


Figure 19

After many number lines and arrows have been drawn, discuss the need for a way of naming the arrows so the children can speak of them and a listener can understand the properties of the arrows without actually seeing them. The question, "How can arrow a be described?" should be answered, "It can be described by the direction of counting and the number of spaces counted."

Tell the class that counting from left to right is considered a positive direction and is indicated by a " $^{+}$ "; counting from right to left is considered a negative direction and is indicated by a " $^{-}$."

Have the children indicate both the direction of counting and the number of spaces counted on the arrows they have just drawn on the board. See Figure 20.

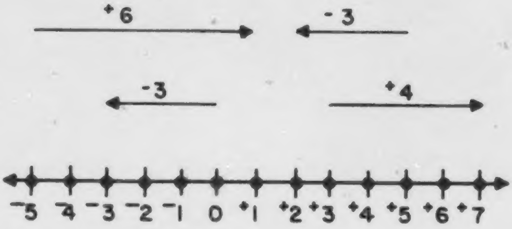


Figure 20

The number of spaces counted is the measure of the arrow. In Figure 20, the arrow representing $^{+}6$ has a measure of 6 (a whole number), and the arrow representing $^{-}3$ has a measure of 3. Remember that the measure is a whole number. The concept of measure is developed further in the next unit.

Pupil pages 201 and 202 are a summary of the material above. Exercise Sets 3 and 4 can now be done independently by the pupils.

ARROW DIAGRAMS

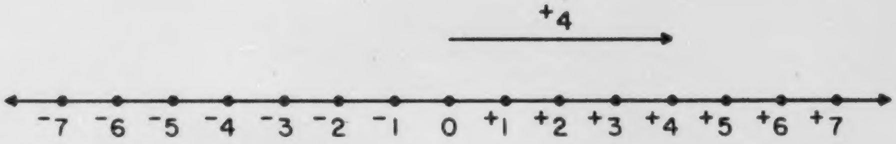
Diagrams may be used to show the result of counting. You can count forward or backward. An arrow can show many things about counting. The arrow in the diagram below shows:

(1) Where you begin to count

(2) The number of spaces you count

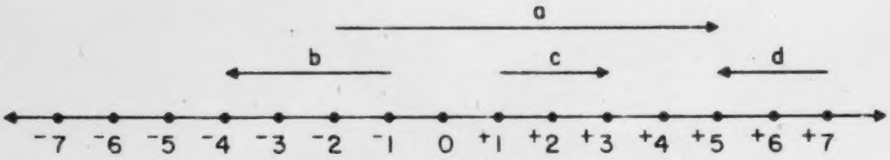
(3) In what direction you count

(4) Where you stop.



In the diagram above, we began at the zero dot and counted 4 in a positive direction to the $^+4$ dot. 4 is called the measure of the arrow. The arrow is named $^+4$ to show the direction of count and the measure. The $^+$ shows direction, 4 shows the measure. The tail of the arrow is at zero; the head of the arrow is at $^+4$.

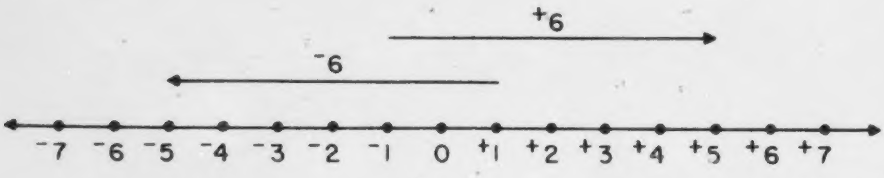
The diagram below shows some other arrows. Name them with integers. (a. $^{+}7$) (b. $^{-}3$) (c. $^{+}2$) (d. $^{-}2$)



We have agreed that counting in a positive direction means counting to the right from a reference dot on the number line. This may be shown by an arrow diagram and labeled with the symbol " + "; $\xrightarrow{\quad ^{+}6 \quad}$ means that you start at a reference dot and count six spaces to the right. The measure of the arrow is 6.

Counting in a negative direction means counting to the left from a reference dot on the number line. The symbol " - " on an arrow diagram shows this; $\xleftarrow{\quad ^{-}6 \quad}$ means that you count six spaces to the left from a reference dot on the number line. The measure of the arrow is 6.

The figure below shows two arrows with the same measure. One indicates a count in the negative direction; the other a count in the positive direction.

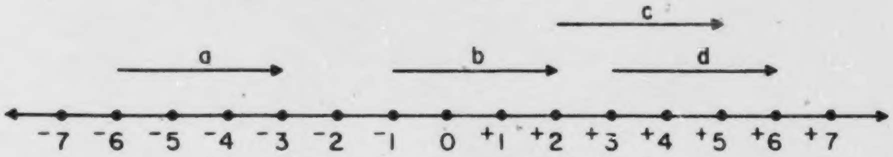


Exercise Set 3

1. Look at the arrows on the number line below.

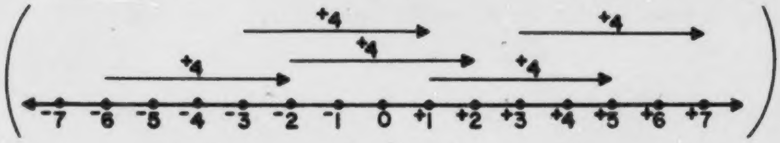
   a. How are they alike? (Each has measure 3 and a positive direction)

   b. How are they different? (They begin and end at different points)



c
a
b
d
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

2. Draw a number line. Label it with integers. Draw five arrows that begin at these dots: ⁺1, ⁻2, ⁻6, ⁺3, ⁻3. ⁺4

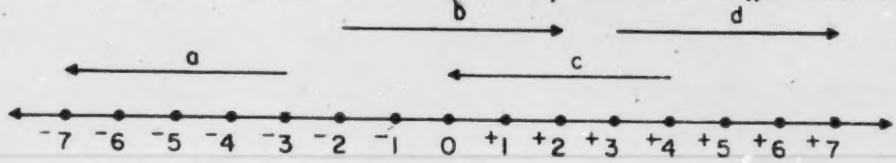


⁺4
⁺4
⁺4
⁺4
⁺4
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

3. Look at the arrows on the number line below.

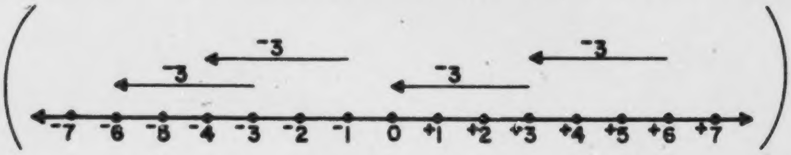
   a. How are they alike? (Each has the measure 4)

   b. How are they different? (They have different directions, positive and negative. They begin and end at different points)



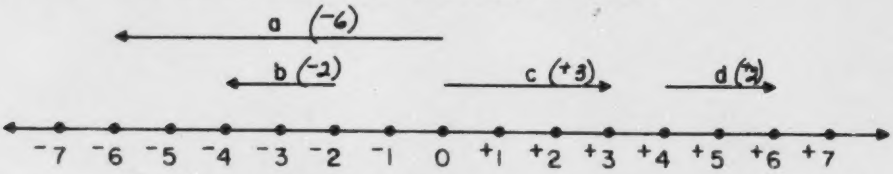
b
d
a
c
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

4. Draw a number line. Label it with integers. Draw four arrows that begin at these dots: ⁻3, ⁺3, ⁺6, ⁻1. ⁻3



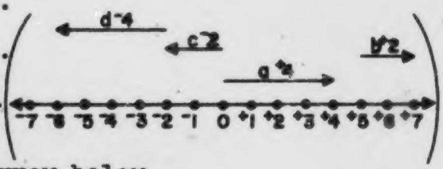
⁻3
⁻3
⁻3
⁻3
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

5. Write the number name for each arrow below.



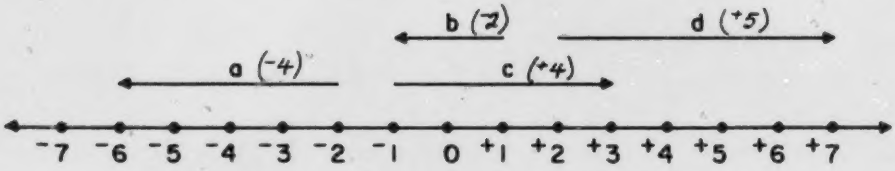
a (⁻6)
b (⁻2)
c (⁺3)
d (⁺2)
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

6. Draw a number line. Label special points with integers. Draw the arrows described below and label them.
   a. Begin at 0 and end at $^{+}4$.
   b. Begin at $^{+}5$ and end at $^{+}7$.
   c. Begin at 0 and end at $^{-}2$.
   d. Begin at $^{-}2$ and end at $^{-}6$.



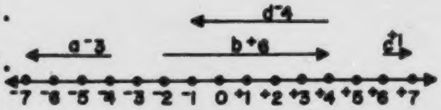
d $^{-}4$
c $^{-}2$
b $^{+}2$
a $^{+}4$
$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$

7. Write the number name for each arrow below.



b ($^{-}2$)
d ($^{+}5$)
a ($^{-}4$)
c ($^{+}4$)
$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$

8. Draw a number line. Label special points with integers. Draw the arrows described below and label them.
   a. Begin at $^{-}4$ and end at $^{-}7$.
   b. Begin at $^{-}2$ and end at $^{+}4$.
   c. Begin at $^{+}6$ and end at $^{+}7$.
   d. Begin at $^{+}3$ and end at $^{-}1$.



d $^{-}4$
a $^{-}3$
b $^{+}6$
c $^{+}1$
$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$

9. Which arrow has the greater measure?
   a. $^{-}2$ to $^{+}6$ or $^{+}2$ to $^{+}6$ ($^{-}2$ to $^{+}6$)
   b. $^{+}8$ to $^{+}1$ or $^{+}8$ to $^{-}1$ ($^{+}8$ to $^{-}1$)
   c. 0 to $^{+}4$ or $^{-}6$ to 0 ($^{-}6$ to 0)
   d. $^{+}5$ to $^{-}3$ or $^{-}3$ to $^{-}5$ ($^{+}5$ to $^{-}3$)
   e. $^{-}4$ to $^{-}8$ or $^{+}6$ to $^{+}9$ ($^{-}4$ to $^{-}8$)

Review

Exercise Set 4

1. At what integer on the number line will you stop, if you

   a. begin at 0 and count 7 to the right? ($^{+}7$)

   b. begin at $^{-}2$ and count 4 to the left? ($^{-}6$)

   c. begin at $^{-}6$ and count 9 to the right? ($^{+}3$)

   d. begin at 0 and count 5 to the left? ($^{-}5$)

   e. begin at $^{+}6$ and count 6 to the left? (0)

   f. begin at $^{+}3$ and count 2 to the right? ($^{+}5$)

   g. begin at $^{-}4$ and count 4 to the right? (0)

2. Draw arrows to represent each answer in exercise 1. Label with an integer.

3. Copy and write the opposite for each integer below. Underline the greater integer in each pair.

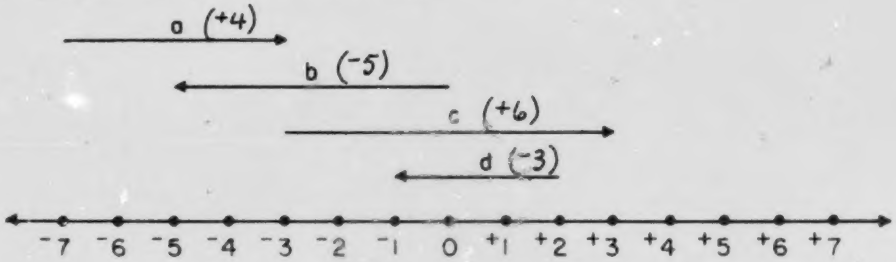
   a. $^{-}7$ $\underline{(^{+}7)}$ c. $\underline{^{+}9}$ $(^{-}9)$ e. $\underline{^{+}6}$ $(^{-}6)$

   b. 0 (0) d. $^{-}4$ $\underline{(^{+}4)}$ f. $^{-}5$ $\underline{(^{+}5)}$

4. Write in words the names of the following integers.

   a. $^{+}9$ (positive nine) b. 0 (zero) c. $^{-}3$ (negative three)

5. Label the arrows below using $^{-}5$, $^{+}4$, $^{-}3$, $^{+}6$.



6. Some of these mathematical sentences are true and some false Mark them T if true and F if false.

a. $^{+}17 > ^{-}99$ (T) e. $^{-}6 > 0$ (F)
b. $^{-}17 > ^{-}99$ (T) f. $^{-}1 > ^{-}6$ (T)
c. $^{-}2 < ^{-}5$ (F) g. $^{+}14 > ^{-}14$ (T)
d. $^{+}2 < ^{+}5$ (T) h. $^{-}14 < ^{+}14$ (T)

7. Choose the largest integer from each set.

		Ans. to Ex. 7	Ans. to Ex. 8
a.	$P = \{^{-}29, ^{+}3, ^{+}31, ^{-}50, ^{-}1\}$	(+31)	(-50)
b.	$T = \{^{+}5, ^{+}1, ^{-}2, ^{-}1, ^{-}5\}$	(+5)	(-5)
c.	$W = \{^{+}23, ^{-}41, ^{-}30, ^{+}29, ^{+}20\}$	(+29)	(-41)
d.	$F = \{0, ^{-}3, ^{-}7, ^{-}2, ^{-}6\}$	(0)	(-7)
e.	$G = \{^{-}4, ^{+}10, ^{+}15, ^{-}9, ^{+}1\}$	(+15)	(-9)

8. Choose the smallest integer from each set in Exercise 7.

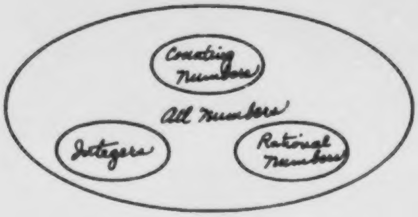
9. Would the arrow drawn for each of the following be named by a positive or negative integer?

a. from $^{-}3$ to $^{-}7$ (negative) c. from $^{-}5$ to $^{-}1$ (positive)
b. from $^{+}2$ to $^{-}4$ (negative) d. from $^{-}4$ to 0 (positive)

10. Name these sets of numbers. The letter used for each set should help you remember the name of the set.

$C = \{1, 2, 3, 4, \ldots\}$ (Counting numbers)
$R = \{\ldots, \frac{1}{2}, \ldots, \frac{4}{4}, \ldots, \frac{17}{8}, \ldots\}$ (Rational Numbers)
$I = \{\ldots, ^{-}2, ^{-}1, 0, ^{+}1, ^{+}2, \ldots\}$ (Integers)

11. Draw a picture diagram like the one below on your paper. Label the sets given in exercise 10.


Counting numbers
All numbers
Integers
Rational numbers

12. Write these subsets of the set of integers.
    a. Integers which are positive ($\{^+1, ^+2, ^+3, ^+4, \cdots\}$)
    b. Integers which are negative ($\{\cdots {}^-4, {}^-3, {}^-2, {}^-1\}$)
    c. Integers which are neither positive nor negative ($\{0\}$)

ADDITION OF INTEGERS

Objective: To use arrow diagrams as an aid in renaming integers like $^{+}5 + {}^{+}2$; $^{-}3 + {}^{-}7$; $^{-}5 + {}^{+}3$; $^{+}4 + {}^{-}1$

To use mathematical sentences to rename integers

To add integers with the aid of an arrow diagram

Materials: Demonstration number lines; (The teacher will find it most helpful to prepare sheets of number lines for the children from a ditto carbon to save time. See page 355.)

Vocabulary: Rename integers

Suggested Teaching Procedure:

> The material in the Mathematical Background, pages 312-325 is useful in this section. The teacher may find it helpful to reread these pages.
>
> Have the children do Exercise Set 5, page 208, as an exploratory lesson.
>
> Place figures 21a, 22a, 23a, and 24a, page 351, on the chalkboard prior to class.

You have drawn arrow diagrams for Exercise Set 5. In each of the exercises you drew three arrows. The second one began where the first one stopped.

Help me describe the diagrams.

In exercise 1 you drew a $^{+}3$ arrow and then began where it stopped and drew a $^{+}5$ arrow. What was the next thing you were asked to do? (Draw an arrow from the tail of the $^{+}3$ arrow to the head of the $^{+}5$ arrow.)

What can we name the dotted arrow? ($^{+}8$)

What does the dotted arrow represent? (The measure and direction of the other two arrows together.)

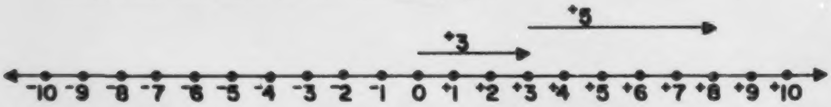
> Repeat with exercise 2.

Exercise Set 5 (Exploratory)

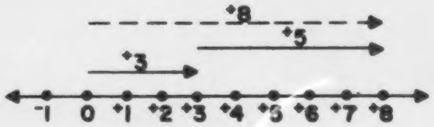
1. Draw a number line. Label it with integers from $^{-}8$ to $^{+}8$.

   a. Begin at zero and count 3 spaces to the right on the number line. Draw an arrow above the number line to show this count. Label it $^{+}3$.

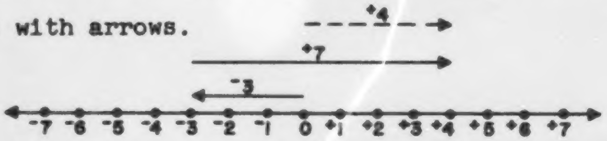
   b. Begin at a point directly above the head of the $^{+}3$ arrow and count 5 more spaces to the right. Draw another arrow to show this count. Label it $^{+}5$. Your diagram should look like this:



   c. Put a finger on zero where you started counting to make the $^{+}3$ arrow. Put your pencil on the dot where the $^{+}5$ arrow ends. What integer does this dot represent? (+8) Above the $^{+}5$ arrow draw a "dotted" arrow, with its tail above the zero, and its head directly above the point where you stopped counting, $^{+}8$. Label the "dotted" arrow with an integer. Your diagram should look like this:



2. a. Change exercise 1 by beginning at zero and counting 3 to the left then 7 to the right. Show both of these counts with arrows.



   b. Draw a "dotted" arrow which begins at zero and ends at the dot where the counting stops. Label the "dotted" arrow with an integer.

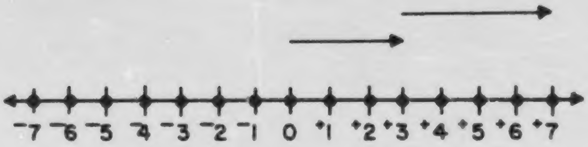

⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

Figure 21a

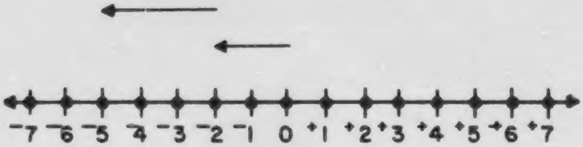

⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

Figure 22a

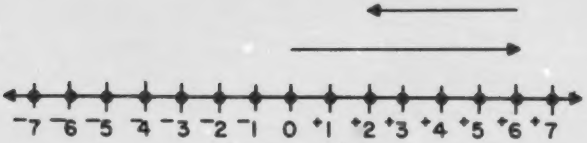

⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

Figure 23a

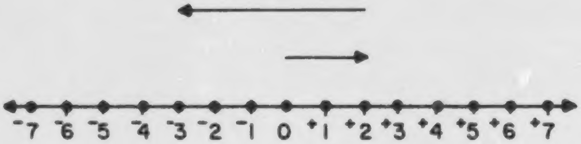

⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

Figure 24a

Refer to Figures 21a, 22a, 23a, and 24a on the chalkboard. The discussion should follow the plan below for each of the figures one at a time. (Figure 21a is used as an example):

(1) Name the arrow which begins at zero. ($^+3$) Where does the arrow end? ($^+3$)

(2) Name the arrow which begins at $^+3$. ($^+4$) Where does the arrow end? ($^+7$)

(3) Draw one arrow with dotted lines to show the measure and direction of both the arrows together. (See Figure 21b)

What do we name this arrow? ($^+7$) We call $^+7$ the sum of $^+3$ and $^+4$. It shows the measure and the direction of the arrow sum.

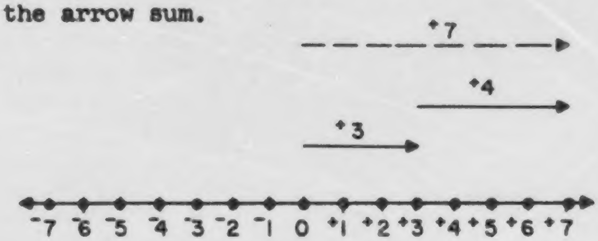

+7
+4
+3
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Figure 21b

A similar discussion of Figures 22a, 23a, and 24a should result in Figure 22a appearing as 22b, Figure 23a as 23b, and Figure 24a as 24b.

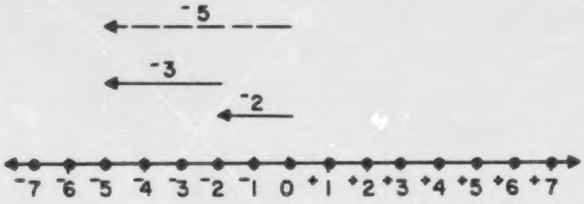

-5
-3
-2
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Figure 22b

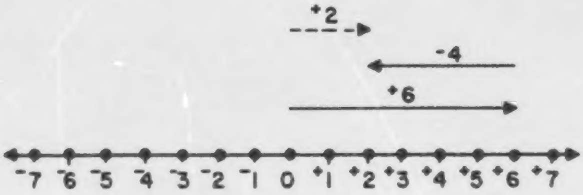

+2
-4
+6
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Figure 23b

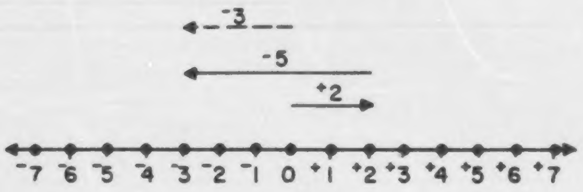

-3
-5
+2
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Figure 24b

Review and extend the ideas developed by the following discussion about Figures 21b, 22b, 23b, and 24b. Direct the discussion with these questions and suggestions (Figure 21b is used as an example):

(1) In each of the diagrams, you first drew two arrows. Name them for Figure 21b. ($^{+}3$ and $^{+}4$)

(2) You showed $^{+}3$ and $^{+}4$ by drawing arrows. Then you drew one "dotted" arrow which showed the sum of the two arrows. Name this arrow. ($^{+}7$)

(3) You have shown the sum of the two integers in two ways with arrows: first, by drawing two arrows, the second one beginning at the point where the first arrow stopped; and then by drawing one dotted arrow which represents the sum of the two arrows. Write a mathematical sentence to show this. ($^{+}3 + {}^{+}4 = {}^{+}7$) The first arrow you drew represents the addend $^{+}3$. The second arrow you drew represents the addend $^{+}4$. The dotted arrow represents the sum.

The mathematical sentences which the children should write for each of the figures are these:

Figure 21b: $^{+}3 + {}^{+}4 = {}^{+}7$

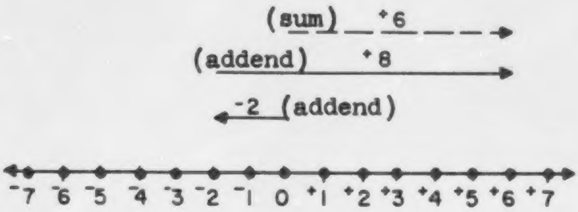
Figure 22b: $^{-}2 + {}^{-}3 = {}^{-}5$

Figure 23b: $^{+}6 + {}^{-}4 = {}^{+}2$

Figure 24b: $^{+}2 + {}^{-}5 = {}^{-}3$

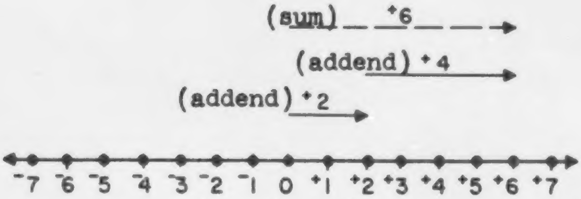
Have the children find several ways of renaming $^{+}6$. They should draw two arrows, the first beginning at zero and the second starting at the point where the first arrow ends and ending at $^{+}6$. The third arrow, which represents the sum, can then be drawn. This should be a dotted arrow extending from zero to the point at which the second arrow stopped. By studying the arrow diagram, the pupil can write the mathematical sentence showing the renaming. See Figures 25, 26, and 27.

Children should understand that each of the first two arrows they draw represents an addend and that the dotted arrow represents the sum, which was unknown. Label the arrows as appropriate, with the words "addend" or "sum." Note that the dotted arrow represents an unknown integer, the sum.


(sum) +6
(addend) +8
-2 (addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

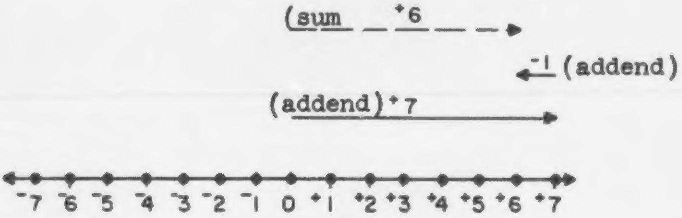
$^{+}6 = {}^{-}2 + {}^{+}8$

Figure 25


(sum) +6
(addend) +4
(addend) +2
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

$^{+}6 = {}^{+}2 + {}^{+}4$

Figure 26


(sum +6
-1 (addend)
(addend) +7
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

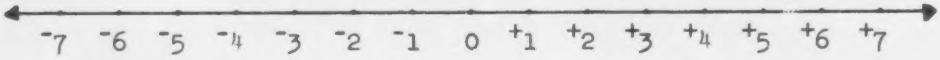
$^{+}6 = {}^{+}7 + {}^{-}1$

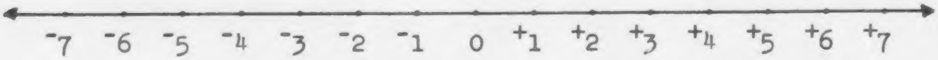
Figure 27

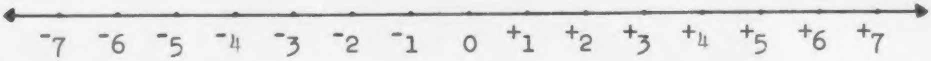
The children should now read the material in their books on "Renaming Integers," page 209 and do the exercises of Set 6 which they need.

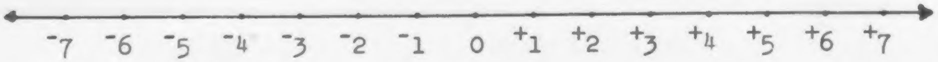
* * *

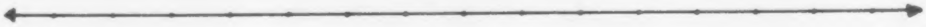
This model sheet may be dittoed by the teacher for the pupils to use in doing the Exercise Sets. One number line is not labeled so that different scales can be written in and used.


$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$


$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$

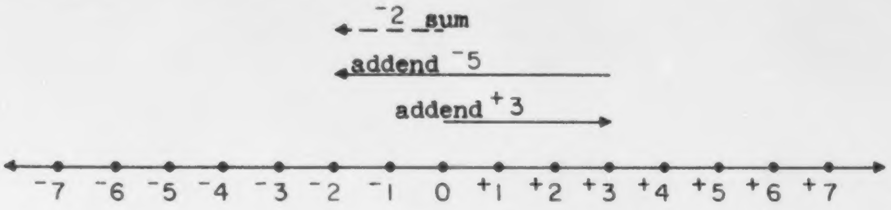

$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$


$^{-}7$ $^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$ $^{+}7$



RENAMING INTEGERS

Arrow diagrams may be used to rename integers. The diagram below renames $^{+}3 + {}^{-}5$ as $^{-}2$. This may be shown by the mathematical sentence $^{+}3 + {}^{-}5 = {}^{-}2$.



The diagram is made by following these steps:

(1) Begin at a point directly above zero and draw a solid arrow for the first addend ($^{+}3$). Draw to the right for positive.

(2) Begin at a point directly above the head of the arrow for the first addend and draw a solid arrow for the second addend ($^{-}5$). Draw to the left for negative.

(3) Above this arrow draw a "dotted" arrow from directly above zero to the head of the arrow for the second addend. This arrow ($^{-}2$) renames $^{+}3 + {}^{-}5$. It is the sum of $^{+}3$ and $^{-}5$.

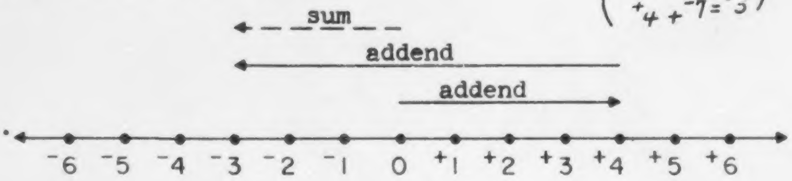
Follow this plan: (1) draw the arrow for the first given addend directly above the number line; its tail should be at 0; (2) draw the arrow for the second addend above the first arrow, starting at the point where the first arrow's head ends; and (3) draw the "dotted" arrow representing the sum above these two arrows. This dotted arrow begins directly above zero and ends at the head of the second arrow. If this plan is followed, we will better understand our diagrams.

Exercise Set 6

1. Study the arrow diagrams below. Write a mathematical sentence for each:

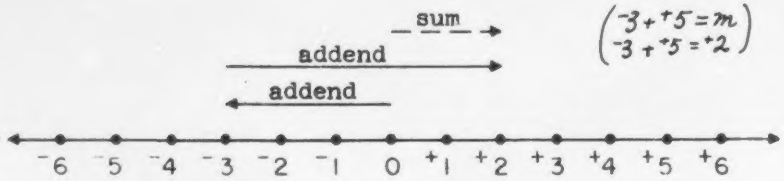
($^+4 + ^-7 = n$
$^+4 + ^-7 = ^-3$)

a.


sum
addend
addend
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

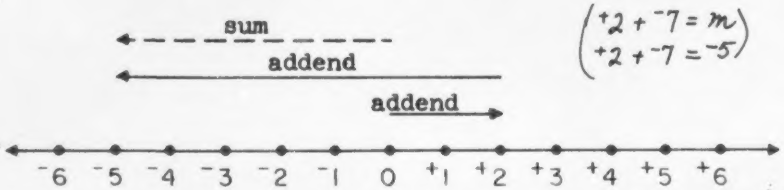
($^-3 + ^+5 = m$
$^-3 + ^+5 = ^+2$)

b.


sum
addend
addend
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

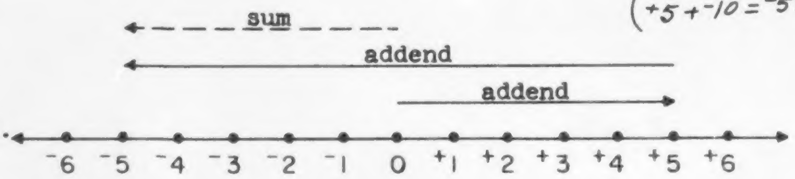
($^+2 + ^-7 = m$
$^+2 + ^-7 = ^-5$)

c.


sum
addend
addend
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

($^+5 + ^-10 = t$
$^+5 + ^-10 = ^-5$)

d.


sum
addend
addend
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

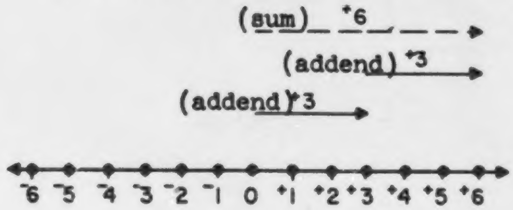
2. Draw a number line and use arrows to illustrate each of these mathematical sentences. (For answers see Teachers' Commentary page 358.)

a. $^+3 + ^+3 = ^+6$
b. $^-2 + ^-1 = ^-3$
c. $^-4 + ^-2 = ^-6$
d. $^+4 + ^-7 = ^-3$
e. $^-5 + ^+9 = ^+4$
f. $^-6 + ^+3 = ^-3$

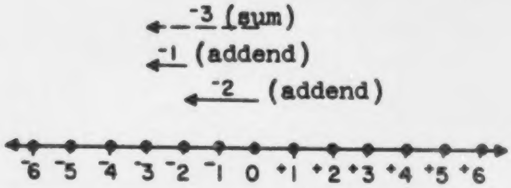
Answers for
exercise 2, Exercise Set 6

2.

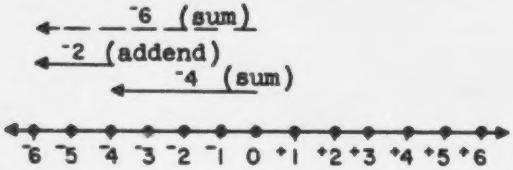
a.


(sum) +6
(addend) +3
(addend) +3
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

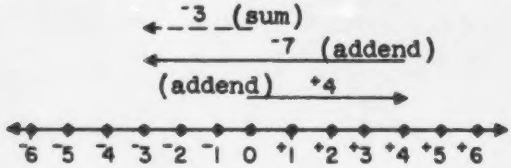
b.


-3 (sum)
-1 (addend)
-2 (addend)
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

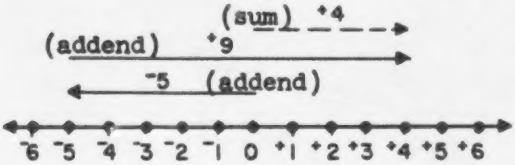
c.


-6 (sum)
-2 (addend)
-4 (sum)
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

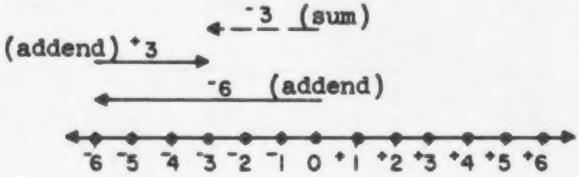
d.


-3 (sum)
-7 (addend)
(addend) +4
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

e.


(sum) +4
(addend) +9
-5 (addend)
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

f.


-3 (sum)
(addend) +3
-6 (addend)
-6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6

3. Rename each integer in Column A by matching it with another name in Column B. You might be able to do this without drawing arrow diagrams. Look at the first sum in Column B. It is $^{+}13 + {}^{-}2$. Can you "imagine" a number line which has the arrows $^{+}13$ and $^{-}2$ on it? The $^{+}13$ arrow would begin directly above zero and have its head directly above $^{+}13$. The $^{-}2$ arrow would begin directly above the head of the $^{+}13$ arrow and would be drawn 2 spaces to the left. Its head would be directly above $^{+}11$. So the arrow for the sum would begin at zero and have its head directly above $^{+}11$. Thus, $^{+}13 + {}^{-}2 = {}^{+}11$; and d is the answer.

A	B
a. $^{+}17$	(d) $({}^{+}13 + {}^{-}2)$
b. $^{-}8$	(e) $({}^{-}20 + {}^{-}4)$
c. $^{+}2$	(a) $({}^{+}9 + {}^{+}8)$
d. $^{+}11$	(f) $({}^{-}5 + {}^{+}2)$
e. $^{-}24$	(b) $({}^{-}5 + {}^{-}3)$
f. $^{-}3$	(c) $({}^{+}1 + {}^{+}1)$

4. Complete with an integer.

a. $^{-}4 + {}^{-}3 = \underline{({}^{-}7)}$

b. $^{+}2 + {}^{-}4 = \underline{({}^{-}2)}$

c. $^{+}3 + {}^{+}5 = \underline{({}^{+}8)}$

d. $^{+}4 + {}^{-}9 = \underline{({}^{-}5)}$

e. $^{-}7 + {}^{+}14 = \underline{({}^{+}7)}$

f. $^{+}8 + {}^{-}3 = \underline{({}^{+}5)}$

g. $^{-}11 + {}^{+}10 = \underline{({}^{-}1)}$

h. $^{-}9 + {}^{-}7 = \underline{({}^{-}16)}$

5. Complete these mathematical sentences.

a. $^{+}8 + (^{-}23) = {}^{-}15$

b. $(^{-}6) + {}^{-}9 = {}^{-}15$

c. $^{+}20 + (^{-}35) = {}^{-}15$

d. $^{+}1 + (^{-}16) = {}^{-}15$

e. $^{-}2 + (^{+}8) = {}^{+}6$

f. $(^{-}9) + {}^{+}15 = {}^{+}6$

g. $(^{+}3) + {}^{+}3 = {}^{+}6$

h. $^{+}9 + (^{-}3) = {}^{+}6$

6. Use an integer from Column B to rename a, b, c, and d in Column A.

A	B
a. $^{-}5 + {}^{-}4$	(d) $^{+}9$
b. $^{+}5 + {}^{-}4$	(a) $^{-}9$
c. $^{-}5 + {}^{+}4$	(b) $^{+}1$
d. $^{+}5 + {}^{+}4$	(c) $^{-}1$

7. Mark true or false.

$^{-}2$ is another name for:

a. $^{+}5 + {}^{-}3$ (F)

b. $^{-}7 + {}^{-}5$ (F)

c. $^{-}8 + {}^{+}6$ (T)

d. $^{+}2 + 0$ (F)

e. $0 + {}^{-}2$ (T)

$^{+}3$ is another name for:

a. $0 + {}^{+}3$ (T)

b. $^{+}9 + {}^{-}12$ (F)

c. $^{-}4 + {}^{+}7$ (T)

d. $^{-}7 + {}^{+}4$ (F)

e. $0 + {}^{-}3$ (F)

8. $^{+}4$ may be renamed, for example, $^{+}1 + {}^{+}3$ or $^{-}1 + {}^{+}5$.

Rename each of the following integers in two different ways. (Answers will vary)

a. $^{+}3$

b. $^{-}4$

c. $^{+}1$

d. 0

e. $^{-}75$

f. $^{+}18$

g. $^{-}100$

h. $^{+}100$

9. **BRAINTWISTER:** Rename each of the following numbers with a numeral.

a. $^{+}5 + ^{-}7 + ^{-}5 + ^{+}7$ (0)

b. $^{-}3 + ^{+}14 + ^{-}4$ $(^{+}7)$

c. $^{-}7 + ^{+}6$ $(^{-}1)$

d. $^{-}137 + ^{+}136$ $(^{-}1)$

e. $^{-}8 + ^{+}8 + ^{-}5 + ^{+}5 + ^{+}1$ $(^{+}1)$

f. $^{+}45 + ^{-}46 + ^{+}1$ (0)

g. $^{-}2 + ^{-}2 + ^{-}2 + ^{-}2$ $(^{-}8)$

h. $^{+}5 + ^{+}5 + ^{-}10$ (0)

Suggested Teaching Procedure (Continued after children have completed Exercise Set 6)

Make sure that the children have an understanding of the addition of integers and can find sums like $^{+}3 + {}^{-}5$; $^{-}4 + {}^{-}3$; $^{-}7 + {}^{+}9$; $^{+}6 + {}^{+}2$; and $^{-}8 + {}^{+}8$ by drawing arrows on a number line and then writing mathematical sentences like $^{+}3 + {}^{-}5 = {}^{-}2$; $^{-}4 + {}^{-}3 = {}^{-}7$; etc. Then you are ready to tell the children that what they have been doing is adding integers and have discovered how to do so themselves.

The children should now read "Renaming Sums" and "Using the Number Line." These may be followed by Exercise Set 7.

* * *

Exercise Set 8 is an exploratory exercise for helping children discover that when opposites are added, the sum is zero. The exploration should be followed by "Opposites," page 220, and Exercise Set 9, page 221.

Exercise Set 10 is an exploratory exercise for helping children discover that in the addition of integers, the order of addends may be changed with no change in the sum. The exploration should be followed by "Order of Addends," page 224, and Exercise Set 11, page 225.

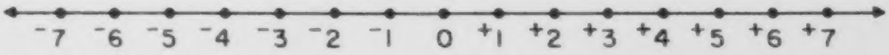
It may be desirable to integrate the exploration of "Opposites" and "Order of Addends" into one development.

RENAMING SUMS

When $^{+}5 + ^{+}2$ is renamed $^{+}7$ from a diagram, when $^{+}5 + ^{-}3$ is renamed $^{+}2$ from a diagram, and when $^{-}2 + ^{-}3$ is renamed $^{-}5$ from a diagram, you are finding sums. The sum of $^{+}5 + ^{+}2$ is $^{+}7$; $^{+}5 + ^{-}3 = ^{+}2$; and $^{-}2 + ^{-}3 = ^{-}5$.

It is not always necessary to draw a diagram. Some of you can look at a number line and imagine the arrows without making them. Try this.

Find the sum in this sentence: $^{-}4 + ^{-}2 = s$. Look at the number line below. **No drawings, please! Imagine the $^{-}4$ arrow,** then the $^{-}2$ arrow. What is the name of the arrow for the sum? It may help to outline the arrows with your eyes or a finger.

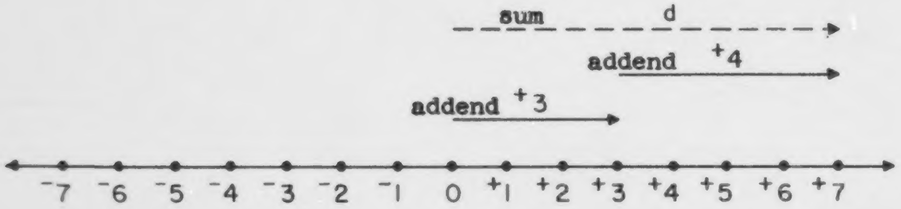


The operation we use when we think of two integers like $^{-}4$ and $^{-}2$ and get $^{-}6$ is called addition. It may be possible for you to add two integers without arrow diagrams or without even looking at a number line. Try it with these: $^{+}3 + ^{+}5$; $^{-}3 + ^{+}2$; $^{-}5 + ^{+}5$.

USING THE NUMBER LINE

The integers and arrow diagrams may be used to solve problems. The diagram below was drawn by a girl to show where a new friend lived. This is the way she explained it:

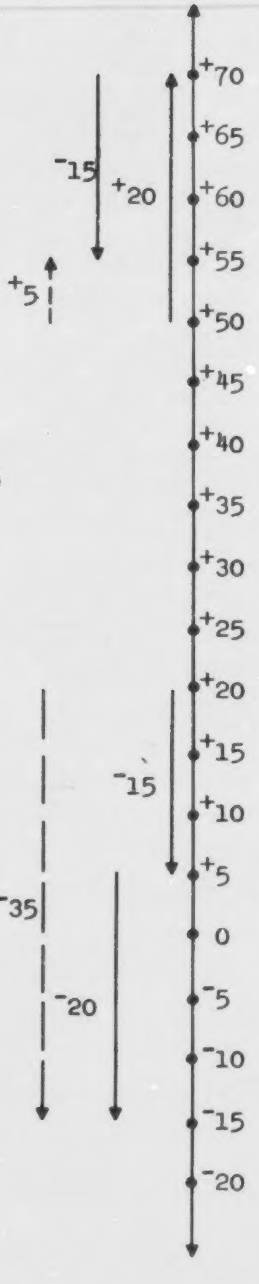
> The line below represents my street and is marked off in blocks. I live at the dot named zero. My friend lives on the same street three blocks east of me. The $^{+}3$ arrow shows this. A new girl has moved in four blocks east of my friend. The $^{+}4$ arrow shows this. The $^{+}7$ arrow shows that the new girl lives 7 blocks east of me.



The diagram above shows the three arrows. The "dotted" arrow shows the sum of the other two arrows. Living seven blocks to the east is the same as living three blocks to the east and then four blocks farther east. This is the meaning of the mathematical sentence which shows addition:

$$^{+}3 + {}^{+}4 = d$$

$$^{+}3 + {}^{+}4 = {}^{+}7$$



The change of temperature as shown by a thermometer may be illustrated by arrow diagrams.

Look at the thermometer scale at the left. It is a vertical number line. It is labeled with integers.

There is an arrow diagram at the top of the thermometer scale which shows a rise of 20° in temperature $(^{+}20)$ and then a fall of 15° $(^{-}15)$. The result of these two changes is shown in the diagram by a "dotted" arrow $(^{+}5^{\circ})$. The mathematical sentence which shows this is $^{+}20 + {}^{-}15 = {}^{+}5$.

The arrow diagram at the bottom of the thermometer scale shows a fall of 15° $(^{-}15)$ in temperature and another fall of 20° $(^{-}20)$. The "dotted" arrow shows the total change in temperature. The mathematical sentence which shows this is $^{-}15 + {}^{-}20 = {}^{-}35$.

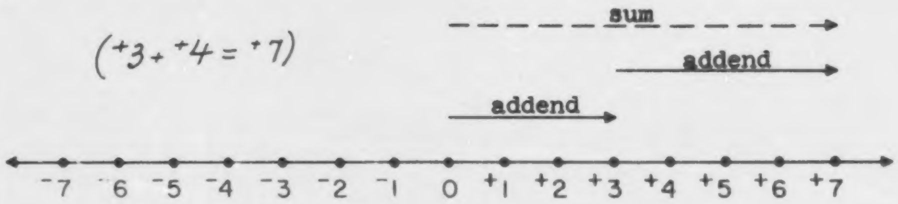
Draw a thermometer scale; sketch in arrows to show two changes. Draw a "dotted" arrow to show the total change shown by the two arrows.

Exercise Set 7

Write a mathematical sentence showing addition for each diagram below.

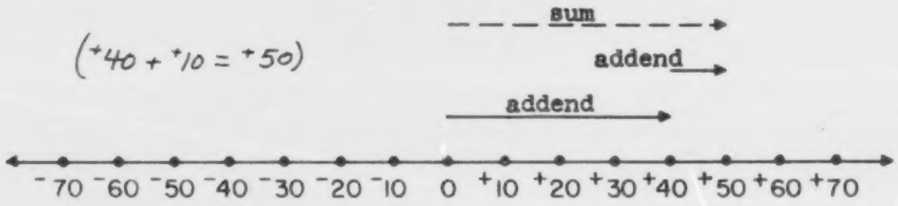
$({}^{+}3 + {}^{+}4 = {}^{+}7)$

1.


sum
addend
addend
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7

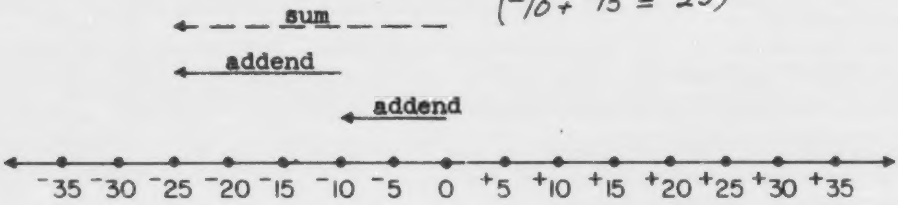
$({}^{+}40 + {}^{+}10 = {}^{+}50)$

2.


sum
addend
addend
⁻70 ⁻60 ⁻50 ⁻40 ⁻30 ⁻20 ⁻10 0 ⁺10 ⁺20 ⁺30 ⁺40 ⁺50 ⁺60 ⁺70

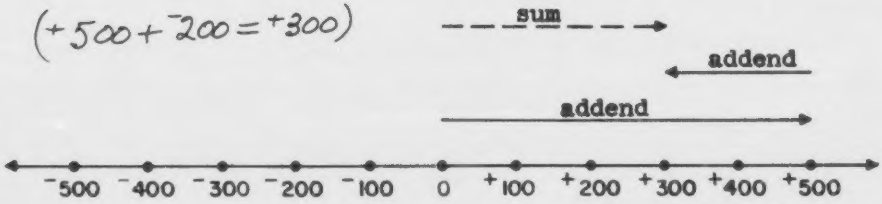
$({}^{-}10 + {}^{-}15 = {}^{-}25)$

3.


sum
addend
addend
⁻35 ⁻30 ⁻25 ⁻20 ⁻15 ⁻10 ⁻5 0 ⁺5 ⁺10 ⁺15 ⁺20 ⁺25 ⁺30 ⁺35

$({}^{+}500 + {}^{-}200 = {}^{+}300)$

4.


sum
addend
addend
⁻500 ⁻400 ⁻300 ⁻200 ⁻100 0 ⁺100 ⁺200 ⁺300 ⁺400 ⁺500

Solve these problems by drawing diagrams if necessary, like those on page 217. Write a mathematical sentence, using integers, for each problem.

5. Charles and John are playing a game. The boy with the greater total score for two games is the winner. Charles has 2 scores. One of them is 5. The other is "3 in the hole." John has 2 scores. One of them is 3. The other is "4 in the hole."
   a. What is Charles' total score? (2)
   b. What is John's total score? ("1 in the hole")
   c. Which boy is the winner? (Charles)

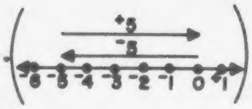
6. The temperature in a mountain cabin was 15 degrees above zero. During the night the temperature fell 20 degrees. What was the temperature then? ($^+15 + {}^-20 = t$, $t = {}^-5$, The temperature was 5 degrees below zero)

7. A scientist invented a "subcopter." (A helicopter which can also go beneath the surface of the water like a submarine.) The "subcopter" was 30 feet above the water. It dived 40 feet. How far below the surface of the water was it then? ($^+30 + {}^-40 = s$, ${}^-10 = s$, It was 10 feet below the surface)

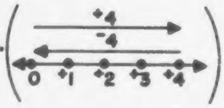
8. Three girls were playing a game. They played it twice. The girl with the smallest total score was the winner. Jane's scores were "5 in the hole" and "8 in the hole." Sandy's scores were "6 in the hole" and 4. Helen's scores were "9 in the hole" and 2.
   a. What was Jane's total score? ("13 in the hole" or ${}^-13$)
   b. What was Sandy's total score? ("2 in the hole, or ${}^-2$)
   c. What was Helen's total score? ("7 in the hole, or ${}^-7$)
   d. Who was the winner? Jane was the winner)
   e. Which girl was second? (Helen was second)

Exercise Set 8

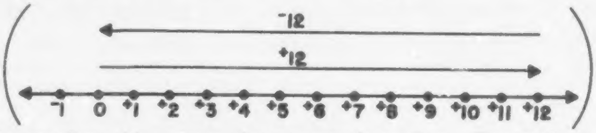
1. Draw an arrow diagram to help rename ${}^{-}5 + {}^{+}5$.



2. Draw an arrow diagram to help rename ${}^{-}4 + {}^{+}4$.



3. Draw an arrow diagram to help rename ${}^{+}12 + {}^{-}12$.



4. What arrow did you draw for the sum in exercise 1? Exercise 2? Exercise 3? (None)

5. How are the answers to exercises 1, 2, and 3 similar?
(They are all zero)

6. What do we call pairs of integers like ${}^{-}5$ and ${}^{+}5$, and ${}^{-}4$ and ${}^{+}4$? (opposites)

7. Choose another pair of opposites and draw an arrow diagram to find its sum. (Answers will vary)

8. Is the sum in exercise 7 the same as those in exercises 1, 2, and 3? (yes)

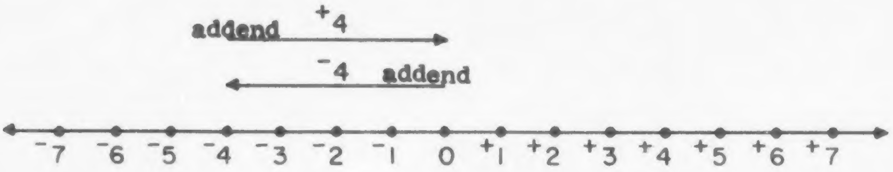
9. What is the sum when opposites are added? (zero)

10. Write a sentence in words about adding opposites.
(When we add opposites the sum is zero.)

OPPOSITES

You were asked to rename $^-5 + ^+5$; $^-4 + ^+4$; and $^+12 + ^-12$ in Exercise Set 8.

The diagram you drew to rename $^-4 + ^+4$ was like the one below. The others were similar.



No arrow was drawn for the new name of $^-4 + ^+4$. You would be drawing an arrow from zero to zero. Counting 4 backward and 4 forward undo each other.

You found: $^-4 + ^+4 = 0$; $^-5 + ^+5 = 0$; and $^+12 + ^-12 = 0$.

$^-4$ and $^+4$ are opposites; $^-5$ and $^+5$ are opposites; $^+12$ and $^-12$ are opposites. We can say:

When opposites are added, the sum is zero.

Exercise Set 9

1. Which of the number pairs below are opposites?

(a, f, h)

a. $^{-}3,\ ^{+}3$

b. $^{-}5,\ ^{-}5$

c. $^{+}7,\ ^{-}2$

d. $^{+}2,\ ^{+}2$

e. $^{-}8,\ 0$

f. $^{+}6,\ ^{-}6$

g. $^{+}1,\ ^{+}1$

h. $^{-}4,\ ^{+}4$

2. Fill the blanks so the sentences are true.

a. $^{+}4 + \underline{(^{-}4)} = 0$

b. $^{-}7 + \underline{(^{+}7)} = 0$

c. $0 = \underline{(^{+}5)} + {}^{-}5$

d. $\underline{(^{+}9)} + {}^{-}9 = 0$

e. $0 = \underline{0} + 0$

f. $^{+}6 + \underline{(^{-}6)} = 0$

3. Which of the following are names for zero?

(c, f, h)

a. $^{+}8 + {}^{+}8$

b. $^{-}6 + 0$

c. $^{-}3 + {}^{+}3$

d. $^{-}4 + {}^{-}4$

e. $^{+}2 + {}^{+}2$

f. $^{-}16 + {}^{+}16$

g. $^{-}7 + {}^{-}7$

h. $^{+}5 + {}^{-}5$

4. Tell whether each of these is a true mathematical sentence. Write "Yes" or "No."

a. $(^{+}2 + {}^{+}4) + (^{-}3 + {}^{-}3) = 0$ (yes)

b. $(^{+}5 + {}^{-}3) + (^{-}5 + {}^{+}3) = 0$ (yes)

c. $(^{-}7 + {}^{+}6) + (^{+}6 + {}^{-}7) = 0$ (no)

d. $(^{+}8 + {}^{-}6) + (^{+}4 + {}^{-}2) = 0$ (no)

e. $(^{-}3 + {}^{+}4) + (^{-}4 + {}^{+}3) = 0$ (yes)

f. $(^{+}9 + {}^{-}12) + (^{-}2 + {}^{-}1) = 0$ (no)

g. $(^{+}26 + {}^{+}5) + (^{-}18 + {}^{-}13) = 0$ (yes)

h. $(^{-}17 + {}^{+}3) + (^{-}2 + {}^{+}16) = 0$ (yes)

5. Which statements are true about 0?
   a. It is neither positive nor negative. (T)
   b. It is equal to its opposite. (T)
   c. It is less than any negative integer.
   d. It is the sum of any integer and its opposite. (T)
   e. It is less than any positive integer. (T)

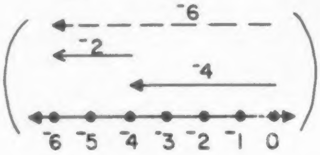
6. Use "positive" or "negative" to complete these sentences.
   a. If an integer is greater than its opposite, the integer is a (positive) integer.
   b. If an integer is less than its opposite, the integer is a (negative) integer.
   c. When you add two negative integers, the sum is a (negative) integer.
   d. When you add two positive integers, the sum is a (positive) integer.
   e. When you add a negative integer and a positive integer, the sum is a (positive) integer if the dot labeled by the positive integer is farther away from 0 than the dot labeled by the negative integer.
   f. When you add a positive integer and a negative integer, the sum is a (negative) integer if the dot labeled by the negative integer is farther away from 0 than the dot labeled by the positive integer.

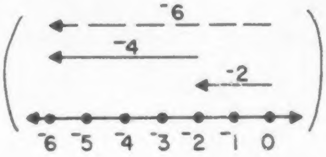
Exercise Set 10

1. Show the addition of these addends on a number line by drawing arrow diagrams.

   a. $^-4 + {}^-2$

   b. $^-2 + {}^-4$





2. Look at your diagrams for **exercise** 1. Answer these questions.

   a. What is the first pair of addends? ($^-4$ and $^-2$)

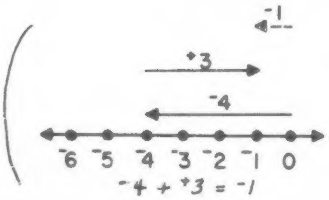
   b. What is the second pair of addends? ($^-2$ and $^-4$)

   c. How are the two pairs of addends alike? (They are the same)

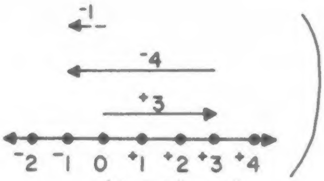
   d. How are the two pairs of addends different? (They are in different order)

   e. What do you notice about the new names you found for the two pairs? (The names are the same.)

3. Rename $^-4 + {}^+3$ and $^+3 + {}^-4$ by drawing arrow diagrams. Answer **exercises** 2a through 2e for these pairs of addends.



$^-4 + {}^+3 = {}^-1$

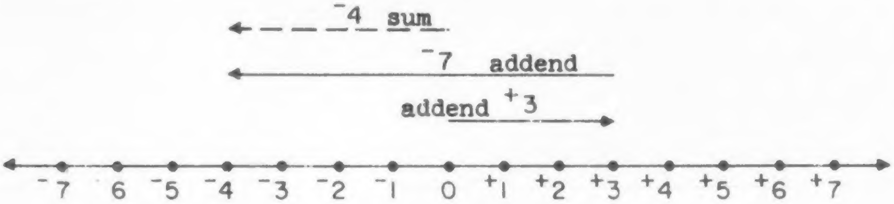


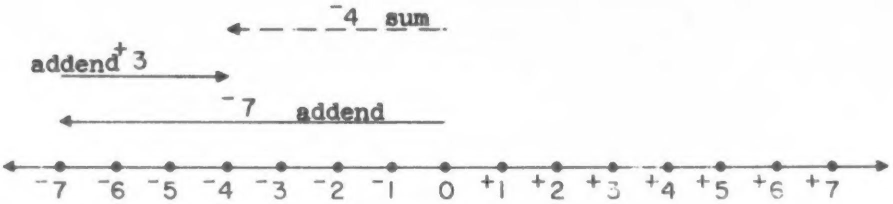
$^+3 + {}^-4 = {}^-1$

ORDER OF ADDENDS

In Exercise Set 10 you drew diagrams to rename $^{-}4 + {}^{-}2$ and $^{-}2 + {}^{-}4$, and to rename $^{-}4 + {}^{+}3$ and $^{+}3 + {}^{-}4$. You found something very interesting.

The diagrams below are similar to ones you drew. They show the renaming of $^{+}3 + {}^{-}7$ and $^{-}7 + {}^{+}3$.


-4 sum
-7 addend
addend +3
-7 6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7


-4 sum
addend +3
-7 addend
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

These diagrams show that $^{+}3 + {}^{-}7$ and $^{-}7 + {}^{+}3$ name the same integer, $^{-}4$.

You found $^{-}4 + {}^{-}2$ and $^{-}2 + {}^{-}4$ each names $^{-}6$; and that $^{-}4 + {}^{+}3$ and $^{+}3 + {}^{-}4$ each names $^{-}1$.

Your work shows that:

> the order of adding two integers may be changed with no change in the sum.
>
> Addition is commutative in the set of integers.

Exercise Set 11

1. Fill the blanks so the sentences are true.

a. ${}^{-}7 + {}^{+}3 = {}^{+}3 + \underline{{}^{-}7}$

b. ${}^{+}14 + {}^{+}8 = \underline{{}^{+}8} + {}^{+}14$

c. $\underline{{}^{-}4} + {}^{-}6 = {}^{-}6 + {}^{-}4$

d. ${}^{+}29 + \underline{{}^{-}12} = {}^{-}12 + {}^{+}29$

e. ${}^{+}51 + {}^{-}5 = \underline{{}^{-}5} + {}^{+}51$

f. $\underline{{}^{+}19} + {}^{-}18 = {}^{-}18 + {}^{+}19$

g. ${}^{-}6 + \underline{{}^{-}17} = {}^{-}17 + {}^{-}6$

h. $\underline{{}^{-}16} + {}^{+}37 = {}^{+}37 + {}^{-}16$

2. Complete these mathematical sentences. The order of adding two addends may be changed without changing the sum.

a. ${}^{+}7 + {}^{-}4 = \underline{{}^{-}4 + {}^{+}7}$

b. $\underline{{}^{-}6 + {}^{-}12} = {}^{-}12 + {}^{-}6$

c. ${}^{+}3 + {}^{+}11 = \underline{{}^{+}11 + {}^{+}3}$

d. $\underline{\qquad\qquad} = {}^{-}26 + {}^{-}7$

e. $\underline{{}^{-}13 + {}^{+}8} = {}^{+}8 + {}^{-}13$

f. ${}^{-}6 + {}^{-}9 = \underline{{}^{-}9 + {}^{-}6}$

g. $\underline{{}^{+}10 + {}^{-}5} = {}^{-}5 + {}^{+}10$

h. ${}^{+}32 + {}^{-}19 = \underline{{}^{-}19 + {}^{+}32}$

3. Complete the mathematical sentences with " >," " <," or " =."

a. ${}^{-}3 + {}^{-}6 \ \underline{(=)}\ {}^{-}6 + {}^{-}3$

b. ${}^{+}3 + {}^{-}6 \ \underline{(<)}\ {}^{-}3 + {}^{+}6$

c. ${}^{+}6 + {}^{+}3 \ \underline{(>)}\ {}^{-}3 + {}^{-}6$

d. ${}^{-}6 + {}^{+}3 \ \underline{(=)}\ {}^{+}3 + {}^{-}6$

e. ${}^{+}7 + {}^{-}2 \ \underline{(<)}\ {}^{+}7 + {}^{+}2$

f. ${}^{+}2 + {}^{-}7 \ \underline{(=)}\ {}^{-}7 + {}^{+}2$

g. ${}^{-}2 + {}^{-}7 \ \underline{(<)}\ {}^{+}2 + {}^{+}7$

h. ${}^{-}2 + {}^{+}7 \ \underline{(>)}\ {}^{-}7 + {}^{+}2$

4. If ${}^{-}6 + {}^{+}2$ is written in each blank below, will the sentence be true or false?

a. $\underline{\qquad\qquad} > {}^{-}8 + {}^{+}8$ (F)

b. $\underline{\qquad\qquad} < 0 + {}^{+}5$ (T)

c. $\underline{\qquad\qquad} > {}^{+}6 + {}^{+}2$ (F)

d. $\underline{\qquad\qquad} > {}^{+}6 + {}^{-}2$ (F)

e. $\underline{\qquad\qquad} < {}^{-}4 + {}^{-}2$ (F)

f. $\underline{\qquad\qquad} < 0 + {}^{-}6$ (F)

g. $\underline{\qquad\qquad} > {}^{+}5 + {}^{-}10$ (T)

h. $\underline{\qquad\qquad} < {}^{-}7 + {}^{-}9$ (F)

5. Write the set whose members will be:

   a. the integers $> {}^{-}4$ and $< {}^{+}2$. $(\{{}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1\})$

   b. the negative integers $> {}^{-}5$. $(\{{}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1\})$

   c. the integers $> {}^{-}3$ and < 0. $(\{{}^{-}2, {}^{-}1\})$

   d. the integers between ${}^{+}2$ and ${}^{-}2$. $(\{{}^{-}1, 0, {}^{+}1\})$

   e. the positive integers $< {}^{+}3$. $(\{{}^{+}2, {}^{+}1\})$

6. Add the following. Use arrow diagrams only when necessary.

   a. ${}^{+}5 + {}^{+}8 = ({}^{+}13)$

   b. ${}^{-}7 + {}^{-}4 = ({}^{-}11)$

   c. ${}^{+}23 + {}^{+}23 = ({}^{+}46)$

   d. ${}^{+}9 + {}^{-}26 = ({}^{-}17)$

   e. ${}^{-}34 + {}^{+}11 = ({}^{-}23)$

   f. ${}^{-}5 + 0 = ({}^{-}5)$

7. In each pair of statements, only one is true. Write the correct statement.

   a. ${}^{+}78 > {}^{-}93$; ${}^{-}78 > {}^{+}93$ $({}^{+}78 > {}^{-}93)$

   b. ${}^{-}15 > {}^{-}2$; ${}^{-}15 < {}^{-}2$ $({}^{-}15 < {}^{-}2)$

   c. ${}^{+}125 < {}^{-}26$; ${}^{-}125 < {}^{+}26$ $({}^{-}125 < {}^{+}26)$

   d. ${}^{+}571 > {}^{-}589$; ${}^{+}571 > {}^{+}589$ $({}^{+}571 > {}^{-}589)$

   e. ${}^{-}2 < {}^{-}35$; ${}^{+}2 < {}^{+}35$ $({}^{+}2 < {}^{+}35)$

   f. ${}^{-}45 > 0$; ${}^{+}45 > 0$ $({}^{+}45 > 0)$

8. An airplane pilot saw that the temperature outside his plane was 23 degrees below zero. A little later, as he was approaching a landing field, he saw that the outside temperature was 40 degrees higher. What was the temperature outside the plane then? (${}^{-}23 + {}^{+}40 = t$, $t = {}^{+}17$
(The temperature was 17 degrees above zero.)

9. The teacher places the end of a pointer on a number line in a sixth grade room. She then moves it along the number line. If it was placed at a point labeled ${}^{+}8$ and moved 9 spaces to the left, at what point did it stop? (${}^{+}8 + {}^{-}9 = t$, $t = {}^{-}1$. It stopped at ${}^{-}1$)

10. These are the scores of three girls on a game.

Betty "6 in the hole."
Mable "9 in the hole."
Ruth "8 in the hole."

On the next game, each girl make a score of 12 points. What is each girl's score then?

(Betty: ${}^{-}6 + {}^{+}12 = {}^{+}6$; Mable: ${}^{-}9 + {}^{+}12 = {}^{+}3$; Ruth ${}^{-}8 + {}^{+}12 = {}^{+}4$)

SUBTRACTION OF INTEGERS

Objective: To subtract integers with the aid of an arrow diagram

To subtract integers by the "short cut" method

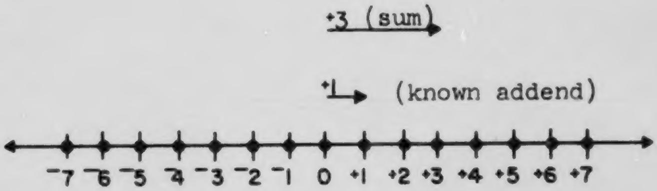
Vocabulary: Undo; unknown addend

Suggested Teaching Procedure:

The following development is for the purpose of helping children draw arrow diagrams to find unknown addends and is dependent on the arrow diagrams used for the addition of two integers.

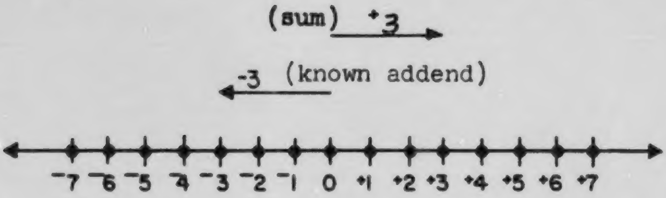
The rereading of the section of the Mathematical Background, pages 320-324 may be found helpful.

Place Figures 28a, 29a, 30a, and 31a on the chalkboard and through the suggested class discussion develop them into Figures 28b, 29b, 30b, and 31b.



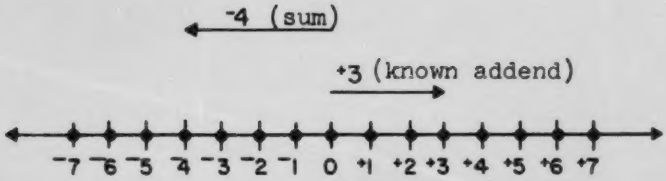
$^{+}1 + n = {}^{+}3$

Figure 28a


(sum) +3
-3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

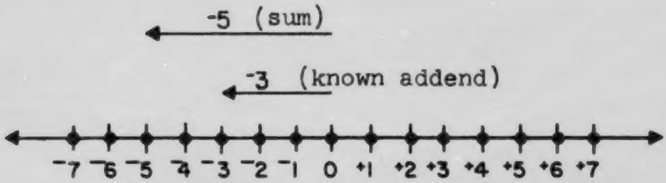
$n + {}^{-}3 = {}^{+}3$

Figure 29a


-4 (sum)
+3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

${}^{+}3 + n = {}^{-}4$

Figure 30a


-5 (sum)
-3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

${}^{-}3 + n = {}^{-}5$

Figure 31a

A discussion of each of the four figures should center around questions and suggestions like those which follow for Figure 28a.

(1) What integers are represented by arrows? (${}^{+}3$ and ${}^{+}1$)

(2) In the sentence, ${}^{+}1 + n = {}^{+}3$, what do these arrows represent? (The ${}^{+}3$ arrow represents the sum and the ${}^{+}1$ arrow represents the known addend.)

(3) What does n represent in the sentence? (The unknown addend)

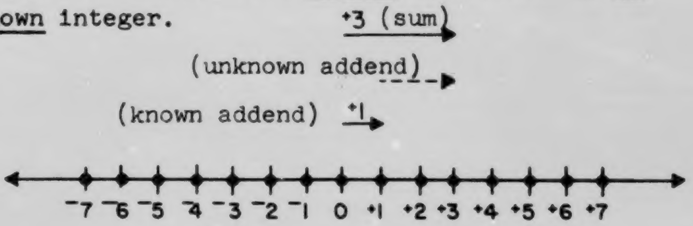
(4) Remembering the diagrams for renaming two addends as a sum, sketch in the arrow for the unknown addend. Put it in as a "dotted" line between the $^+3$ arrow and the $^+1$ arrow. (See Figure 28b. The arrow for the unknown addend should be drawn from the head of the known addend arrow to the head of the sum arrow.)

(5) Where does the arrow you sketched for the unknown addend begin and end? (It begins at $^+1$ and ends at $^+3$.)

(6) Label it by direction and measure. ($^+2$)

(7) What is the unknown addend in the mathematical sentence, $^+1 + n = {}^+3$? ($n = {}^+2$)

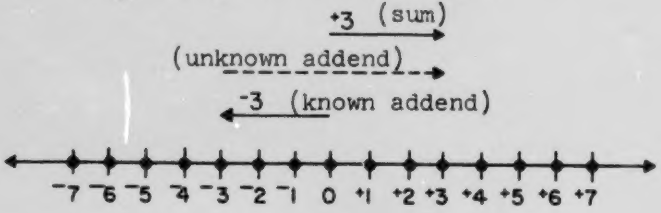
Figure 28a becomes Figure 28b as the children draw in the arrow for the unknown addend. Review the final form of Figure 28b. The children should note the similarity of the diagrams for finding the sum in $^+1 + {}^+2 = s$ and the unknown addend in $^+1 + n = {}^+3$. Help children see that the arrow for the known addend is drawn first and then the arrow is drawn for the sum. In $n + {}^-3 = {}^+3$, the known addend is $^-3$ so its arrow is drawn first. The arrow for the sum, $^+3$, is drawn next. The arrow for the unknown addend is drawn last--just as the arrow for the unknown sum was drawn last. This is a dotted arrow and represents an unknown integer. Thus, the dotted arrow always represents an unknown integer.



$^+1 + n = {}^+3; \quad n = {}^+2$

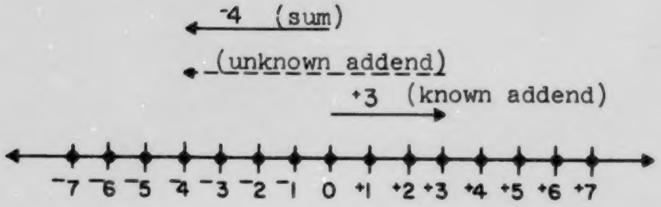
Figure 28b

Similar discussions of Figures 29a, 30a, and 31a should follow and the arrows representing the unknown addends drawn. The figures then appear as Figures 29b, 30b, and 31b.


+3 (sum)
(unknown addend)
-3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

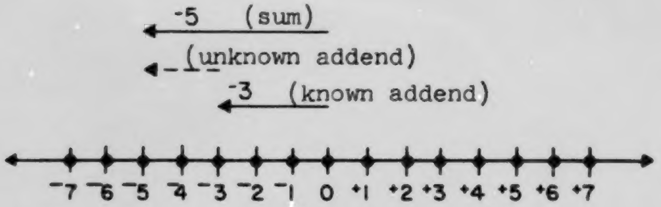
$n + {}^{-}3 = {}^{+}3$; $n = {}^{+}6$

Figure 29b


-4 (sum)
(unknown addend)
+3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

${}^{+}3 + n = {}^{-}4$; $n = {}^{-}7$

Figure 30b


-5 (sum)
(unknown addend)
-3 (known addend)
-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

${}^{-}3 + n = {}^{-}5$; $n = {}^{-}2$

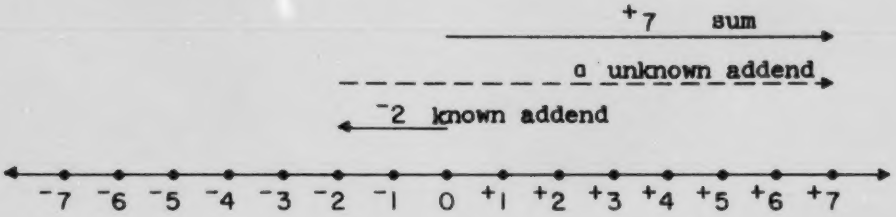
Figure 31b

A discussion of the comparison of the arrow diagrams for finding a sum and one for finding a missing addend should follow.

When the use of arrow diagrams to find an unknown addend is understood by the children, they should read "Introduction to Unknown Addends", page 228 and do Exercise Set 12. Time will be saved if dittoed copies of number lines are provided.

INTRODUCTION TO UNKNOWN ADDENDS

Diagrams may be used to rename a sum when one addend is unknown. If the sum is $^{+}7$ and one addend is $^{-}2$, the mathematical sentence is $^{+}7 = {}^{-}2 + a$ or $^{-}2 + a = {}^{+}7$. The diagram below renames $^{+}7$ as $^{-}2$ and the number represented by the "dotted" arrow.



The arrow for the known addend, $^{-}2$, is drawn directly above the number line. It is a solid line because it is a known addend. Notice that the arrow representing the sum, $^{+}7$, is the top arrow. It is a solid line because it is a known sum.

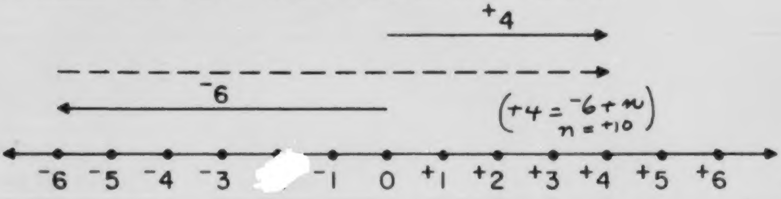
The arrow representing the unknown addend is drawn as a "dotted" line between the other two arrows. It must be drawn so that the sum of it and the arrow representing the known addend is the sum arrow, $^{+}7$.

The arrow for the unknown addend is drawn from the head of the arrow representing the known addend to the head of the arrow representing the sum. In this sentence the arrow represents $^{+}9$.

Exercise Set 12

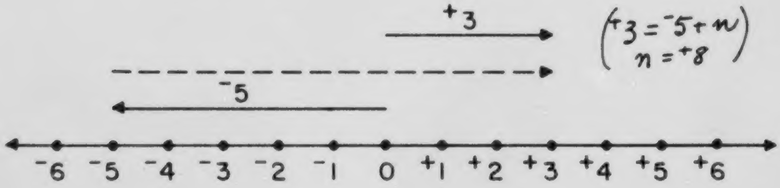
1. Study the arrow diagrams below. Write a mathematical sentence to rename the sum. Find the unknown addend.

a.


$^{+}4$
$^{-}6$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$

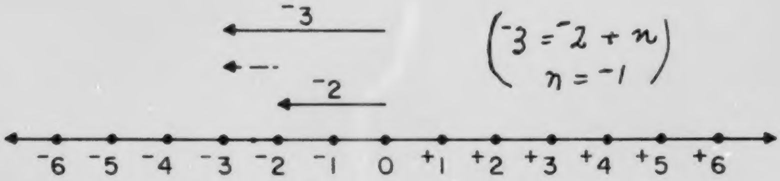
$\left({}^{+}4 = {}^{-}6 + n\right.$
$\left. n = {}^{+}10\right)$

b.


$^{+}3$
$^{-}5$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$

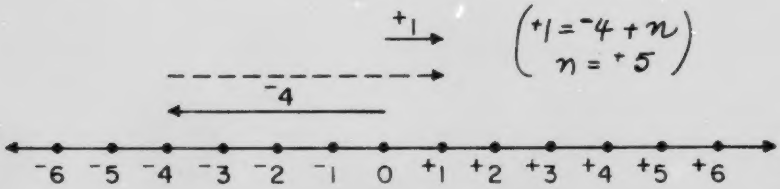
$\left({}^{+}3 = {}^{-}5 + n\right.$
$\left. n = {}^{+}8\right)$

c.


$^{-}3$
$^{-}2$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$

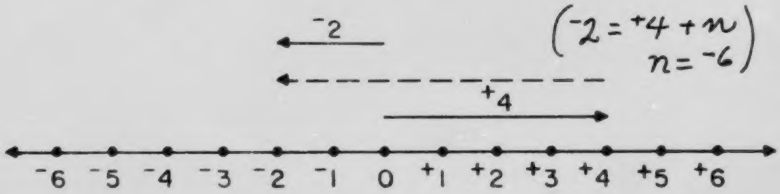
$\left({}^{-}3 = {}^{-}2 + n\right.$
$\left. n = {}^{-}1\right)$

d.


$^{+}1$
$^{-}4$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$

$\left({}^{+}1 = {}^{-}4 + n\right.$
$\left. n = {}^{+}5\right)$

e.


$^{-}2$
$^{+}4$
$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 $^{+}1$ $^{+}2$ $^{+}3$ $^{+}4$ $^{+}5$ $^{+}6$

$\left({}^{-}2 = {}^{+}4 + n\right.$
$\left. n = {}^{-}6\right)$

2. What integer in Column A may be used to complete each sentence in Column B?

A	B
a. $^{+}9$	a. $(-12) + {}^{-}6 = {}^{-}18$
b. $^{-}12$	b. $^{+}18 + (-7) = {}^{+}11$
c. $^{+}3$	c. $^{-}4 + (+6) = {}^{+}2$
d. $^{-}7$	d. $(-2) + {}^{+}5 = {}^{+}3$
e. $^{+}6$	e. $^{-}9 + (+3) = {}^{-}6$
f. $^{-}2$	f. $^{+}5 + (+9) = {}^{+}14$

3. Complete the following sentences.

a. $^{+}7 + (-3) = {}^{+}4$

b. $(-8) + {}^{+}6 = {}^{-}2$

c. $(+14) + {}^{-}8 = {}^{+}6$

d. $^{-}9 + (+12) = {}^{+}3$

e. $(-9) + {}^{+}3 = {}^{-}6$

f. $^{-}4 + (+7) = {}^{+}3$

g. $^{+}8 + (-8) = 0$

h. $(-11) + {}^{+}9 = {}^{-}2$

4. Rename the integers below by completing the mathematical sentences.

a. $^{+}3 + (+6) = {}^{+}9$

b. $(+18) + {}^{-}9 = {}^{+}9$

c. $^{-}15 + (+24) = {}^{+}9$

d. $^{-}1 + (+10) = {}^{+}9$

e. $^{+}4 + (-16) = {}^{-}12$

f. $(-30) + {}^{+}18 = {}^{-}12$

g. $(-6) + {}^{-}6 = {}^{-}12$

h. $^{-}17 + (+5) = {}^{-}12$

5. Diagram each of these mathematical sentences to find the unknown addend.

a. ${}^{-}5 + m = {}^{+}3$ $({}^{+}8)$

b. $n + {}^{-}1 = {}^{+}6$ $({}^{+}7)$

c. $r + {}^{+}4 = {}^{-}7$ $({}^{-}11)$

d. ${}^{+}2 + s = {}^{-}3$ $({}^{-}5)$

e. ${}^{+}7 + t = {}^{+}10$ $({}^{+}3)$

f. ${}^{-}2 + p = {}^{-}9$ $({}^{-}7)$

6. Column A represents temperatures at 6:00 a.m. Column B represents temperatures at 4:00 p.m. Find the total change in temperature between 6:00 a.m. and 4:00 p.m. Use an integer to indicate the amount and direction of change.

	A	B	
Monday	${}^{-}5^{\circ}$	${}^{+}2^{\circ}$	$({}^{+}7^{\circ})$
Tuesday	${}^{+}8^{\circ}$	${}^{-}4^{\circ}$	$({}^{-}12^{\circ})$
Wednesday	${}^{+}3^{\circ}$	0°	$({}^{-}3^{\circ})$

7. Below is a list of cities and the location of each. We think of directions north of the equator as positive.

Hilo	20° north
Rio de Janeiro	23° south
San Francisco	38° north
Lima	12° south

Complete:

a. San Francisco is (61) degrees north of Rio de Janeiro.

b. Lima is (32) degrees south of Hilo.

c. Rio de Janeiro is (43) degrees south of Hilo.

d. Lima is (50) degrees south of San Francisco.

Suggested Teaching Procedure (Continued):

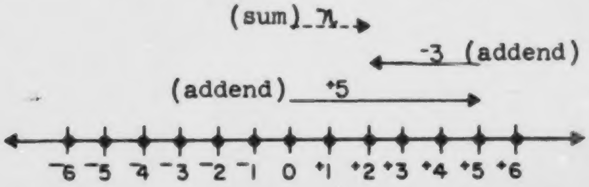
> After Exercise Set 12 has been finished and those exercises discussed which need to be discussed, a summary of the method of finding an unknown addend is recommended. Suggestions for this are (write these two mathematical sentences on the chalkboard: ${}^{+}5 + {}^{-}3 = n$ and ${}^{+}5 + n = {}^{-}2$):

(1) How have you been finding n in sentences like ${}^{+}5 + {}^{-}3 = n$? (By diagraming the sentences with arrows on a number line)

(2) In the sentence ${}^{+}5 + {}^{-}3 = n$, what operation is indicated? (Addition) What are ${}^{+}5$, ${}^{-}3$, and n called in the addition sentence? (${}^{+}5$ and ${}^{-}3$ are addends. n is the sum.)

(3) What are you asked to find in the sentence? (The sum)

(4) Diagram the sentence. (See Figure 32) What integer does n name? (${}^{+}2$)



${}^{+}5 + {}^{-}3 = n$ ${}^{+}5 + {}^{-}3 = {}^{+}2$

Figure 32

(5) You have renamed ${}^{+}5 + {}^{-}3$ as ${}^{+}2$. You have been adding integers. You have also been finding n in sentences like ${}^{+}5 + n = {}^{-}2$. How did you find the integer n? (By diagraming the sentences with arrows on a number line)

(6) In the sentence $^{+}5 + n = {}^{-}2$, give the names for $^{+}5$, for n, and for $^{-}2$ in the sentence. ($^{+}5$ and n are addends. $^{+}5$ is the known addend and n is the unknown addend. $^{-}2$ is the sum.)

(7) What are you asked to find in the sentence? (The unknown addend)

(8) Diagram the sentence. (See Figure 33) What integer does n name? ($^{-}7$)

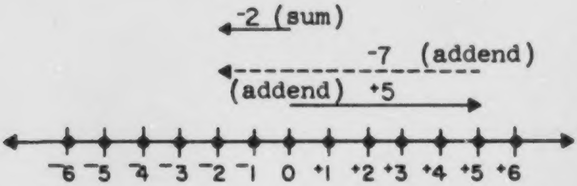


Figure 33

(9) How is the sum $^{-}2$ renamed by the diagram? ($^{+}5 + {}^{-}7$)

(10) By renaming the sum you found the unknown addend. What operation is used to find an unknown addend? (Subtraction)

(11) Write the addition sentence $^{+}5 + n = {}^{-}2$ as a subtraction sentence. ($n = {}^{-}2 - {}^{+}5$)

The teacher should remember that the aim of this unit is to develop vocabulary and meaning for addition and subtraction of integers by the use of diagrams. Mastery is not expected or desired at this time. Each child should be able to add and subtract integers using an arrow diagram. Some children will be able to add and subtract integers without a number line; however, do not require this of all students.

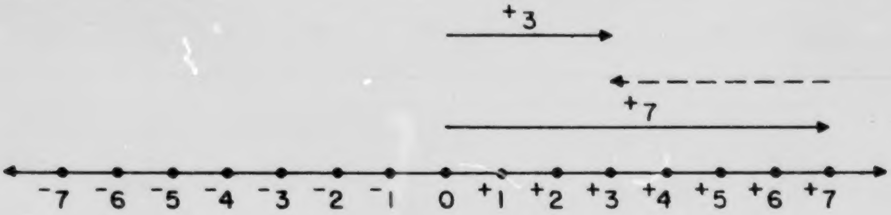
The children should now read "Finding an Unknown Addend", page 232, and do Exercise Set 13, page 233.

FINDING AN UNKNOWN ADDEND

The process of finding an unknown addend in a sentence like $^{-}5 + n = {}^{-}7$ is subtraction.

The diagram on page 228 renamed $^{+}7$ as $^{-}2 + {}^{+}9$. It helped you find an unknown addend which is the result when you subtract $^{-}2$ from $^{+}7$.

The sentence $n + {}^{+}7 = {}^{+}3$ is diagramed below and the steps outlined.



(1) **A solid arrow for the known addend, $^{+}7$, is drawn.**

(2) **A solid arrow for the sum, $^{+}3$, is drawn.**

(3) A "dotted" arrow for the unknown addend, n, is drawn from the head of the $^{+}7$ arrow to the head of the $^{+}3$ arrow.

(4) The "dotted" arrow is named $^{-}4$.

The subtraction of integers may be shown by drawing arrow diagrams. To show subtraction by the use of arrow diagrams, you must find an arrow to represent an unknown addend.

Exercise Set 13

1. Find the unknown addends. (Use an arrow diagram.)

   a. ${}^{-}4 + (\underline{{}^{+}7}) = {}^{+}3$

   b. $(\underline{{}^{+}3}) + {}^{-}10 = {}^{-}7$

   c. ${}^{+}16 + (\underline{{}^{+}18}) = {}^{+}34$

   d. ${}^{+}12 + (\underline{{}^{-}9}) = {}^{+}3$

   e. ${}^{-}7 + (\underline{{}^{+}5}) = {}^{-}2$

   f. $(\underline{{}^{-}17}) + {}^{+}12 = {}^{-}5$

   g. ${}^{-}7 + (\underline{{}^{+}13}) = {}^{+}6$

   h. $(\underline{{}^{+}5}) + {}^{-}15 = {}^{-}10$

2. **Use arrow diagrams to find each unknown addend below.**

   a. ${}^{-}5 + (\underline{{}^{+}9}) = {}^{+}4$

   b. $(\underline{{}^{+}5}) + {}^{-}7 = {}^{-}2$

   c. ${}^{+}2 + (\underline{{}^{+}4}) = {}^{+}6$

   d. $(\underline{{}^{+}5}) + {}^{-}5 = 0$

   e. $(\underline{{}^{+}5}) + {}^{-}2 = {}^{+}3$

   f. ${}^{+}7 + (\underline{{}^{-}10}) = {}^{-}3$

   g. ${}^{-}5 + (\underline{{}^{+}12}) = {}^{+}7$

   h. $(\underline{{}^{-}7}) + {}^{+}3 = {}^{-}4$

3. **Diagram on a number line**

   a. **Two trains started from the same station but traveled in opposite directions.** Train A traveled north at **the rate of** 46 **miles per hour.** Train B traveled south at the rate of 53 miles per hour. How far north of train B would train A be at the end of the first hour?
   (Train A would be 99 miles north of train B)

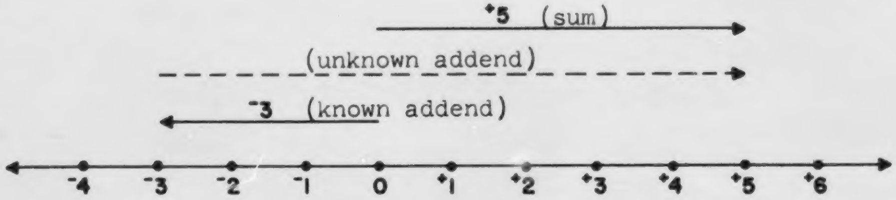
   b. **In a game, Jane's score was** 56 **and Mary's score was "23 in the hole." How many points was Jane ahead of Mary?**
   (Jane was 79 points ahead of Mary)

The "Short Cut" Method For Subtracting Integers

Suggested Teaching Procedure:

The children should now be able to recognize a short cut for finding an unknown addend, i.e., for a subtraction. Some of them may have made the observations to which you should now direct their attention. It will be important to observe again that subtraction is finding an unknown addend and that, for example $^{+}5 - {}^{-}3 = n$ has the same meaning as $^{+}5 = n + {}^{-}3$.

In finding the integer represented by n in $^{+}5 - {}^{-}3 = n$ we are finding the unknown addend in $^{+}5 = n + {}^{-}3$. We know how to make the arrow diagram.



Is the arrow that represents the unknown addend drawn from a point just above the head of the arrow representing the known addend? (Yes.)

To where is it drawn? (To a point directly below the head of the arrow representing the sum)

Which one of the symbols " + " and " - " can we use to denote the direction in which it is drawn? (+)

What is the measure of the length of the arrow which represents the unknown addend? (8)

The number 8 gives us the measure of the arrow. The " + " denotes the direction in which it is drawn.

Can you see an easy way to tell what the unknown addend is? (Yes, it is $^{+}8$. We use 8 to mean the arrow has measure 8, and $^{+}8$ to mean the arrow is drawn 8 spaces to the right.) We can write $^{+}5 - {}^{-}3 = {}^{+}8$; also $^{+}5 = {}^{+}8 + {}^{-}3$.

This type of discussion should follow several exercises in finding unknown addends by arrow diagrams. The children should have acquired considerable skill in making the diagrams. The short cut to which you are now leading is the following: In finding the unknown addend in an addition problem, count the number of spaces from the head of the arrow that represents the known addend to the head of the arrow that represents the sum. This number is the measure of the arrow in spaces. The direction in which you count determines the superscript symbol for the numeral which names the number of spaces in the length of the arrow. The superscript is positive if you counted toward the right and negative if you counted toward the left. This integer is the unknown addend.

Example: $^{+}7 + n = {}^{-}6$. Count from the point labeled $^{+}7$ on the number line to the point labeled $^{-}6$ on the number line. There were 13 spaces counted. The counting was toward the left. Then the unknown addend is $^{-}13$. We write $^{+}7 + {}^{-}13 = {}^{-}6$; also, $^{-}6 - {}^{+}7 = {}^{-}13$. Precision in language is not expected of the pupils at this stage. It would be better to accept "Count from $^{+}7$ to $^{-}6$; the count is 13; the count is to the left: the addend is $^{-}13$; $^{-}6 - {}^{+}7 = {}^{-}13$," than to insist on more precise wording and have the pupils lost in a linguistic maze.

In the sentence $^{+}5 + n = {}^{+}3$, count from the known addend ($^{+}5$) to the sum ($^{+}3$). Two spaces are counted. The counting is toward the left so the unknown addend is $^{-}2$.

Some degree of proficiency might be expected in use of the "short cut in subtraction" or "short cut in finding an unknown addend" before the pupils are asked to do Exercise Set 14. No exposition is given to the pupils in their book preceding this Exercise Set. It will be better for you to develop the "short cut" with demonstrations on the number line and by leading the pupils in class discussion than to have the pupils read an exposition which at the briefest would be very long.

Some pupils will quickly understand that they need not "count" to get the answer. They will in the example $^+7 + n = {}^-6$, think something like this, "To go from $^+7$ to $^-6$, I go from $^+7$ to 0. This is 7 spaces to the left. Then I go from 0 to $^-6$. This is six spaces to the left. I have gone $7 + 6$ or 13 spaces to the left so my answer is $^-13$." For $^+5 + n = {}^+3$, a pupil might think, "I go from $^+5$ to $^+3$. This is $5 - 3$ or 2 spaces to the left. So my answer is $^-2$." Our goal at this stage is not to have all the pupils thinking in a rather sophisticated manner, but certainly any such observations by the pupils should be encouraged.

The "Short-Cut" Method For Subtracting Integers

Exercise Set 14

1. One addend in each of the following is represented by n. What integer does n represent in each?

a. $^{+}2 + n = {}^{+}6$ (+4)
b. $0 + n = {}^{+}6$ (+6)
c. $^{-}2 + n = {}^{+}6$ (+8)
d. $^{-}4 + n = {}^{+}6$ (+10)
e. $^{-}6 + n = {}^{+}6$ (+12)
f. $^{+}7 + n = {}^{-}2$ (-9)
g. $^{+}5 + n = {}^{-}2$ (-7)
h. $^{+}3 + n = {}^{-}2$ (-5)
i. $^{+}1 + n = {}^{-}2$ (-3)
j. $^{-}1 + n = {}^{-}2$ (-1)

2. What addend is represented by n in each sentence?

a. $n + {}^{-}5 = {}^{-}10$ (-5)
b. $n + {}^{-}3 = {}^{-}10$ (-7)
c. $n + {}^{-}1 = {}^{-}10$ (-9)
d. $n + {}^{+}1 = {}^{-}10$ (-11)
e. $n + {}^{+}3 = {}^{-}10$ (-13)
f. $n + {}^{+}5 = {}^{-}10$ (-15)

3. Find the unknown addend.

a. $^{+}12 = n + {}^{+}4$ (+8)
b. $^{-}7 = {}^{-}3 + n$ (-4)
c. $^{-}8 = n + {}^{+}6$ (-14)
d. $^{+}9 = {}^{+}2 + n$ (+7)
e. $^{+}14 = n + {}^{-}2$ (+16)
f. $^{-}2 = {}^{-}1 + n$ (-1)

4. What integer must be added to each of the following to obtain a sum of $^{+}6$?

a. $^{+}3$ (+3)
b. $^{-}3$ (+9)
c. 0 (+6)
d. $^{+}9$ (-3)
e. $^{-}9$ (+15)
f. $^{+}6$ (+3)

5. **Write a true mathematical sentence using addition and these integers:**

a. $^{-}6$, $^{+}8$, $^{+}14$ Answer: $^{+}8 = {}^{-}6 + {}^{+}14$

b. $^{+}5$, $^{-}3$, $^{-}8$ $({}^{-}3 = {}^{+}5 + {}^{-}8)$

c. $^{-}4$, $^{+}2$, $^{-}6$ $(-4 = {}^{+}2 + {}^{-}6)$

d. $^{+}6$, $^{+}3$, $^{+}9$ $(+9 = +6 + {}^{+}3)$

e. $^{-}6$, $^{+}3$, $^{-}9$ $(-6 = {}^{+}3 + {}^{-}9)$

f. $^{-}6$, $^{+}3$, $^{+}9$ $({}^{+}3 = {}^{+}9 + {}^{-}6)$

WRITING SUBTRACTION SENTENCES

Subtraction is the operation of finding an unknown addend. Sentences like these have unknown addends.

$$n + {}^{-}5 = {}^{+}8 \qquad {}^{+}6 = n + {}^{+}2$$

$$ {}^{-}2 + n = {}^{-}5 \qquad {}^{+}8 = {}^{+}12 + n$$

To find n, the known addend is subtracted from the sum. To find n in $n + {}^{-}5 = {}^{+}8$, ${}^{-}5$ is subtracted from ${}^{+}8$.

To show this in a mathematical sentence you may write

$$n = {}^{+}8 - {}^{-}5$$

${}^{-}2 + n = {}^{-}5$ may be written as $n = {}^{-}5 - {}^{-}2$

${}^{+}6 = n + {}^{+}2$ may be written as $n = {}^{+}6 - {}^{+}2$

${}^{+}8 = {}^{-}12 + n$ may be written as $n = {}^{+}8 - {}^{-}12$

Exercise Set 15

1. Write the following subtraction sentences as addition sentences.

a. $^{-}4 - ^{+}3 = s$ $(^{-}4 = s + ^{+}3)$

b. $^{-}9 - ^{+}12 = i$ $(^{-}9 = i + ^{+}12)$

c. $^{-}7 - ^{-}9 = x$ $(^{-}7 = x + ^{-}9)$

d. $^{+}27 - ^{+}25 = t$ $(^{+}27 = t + ^{+}25)$

e. $^{-}35 - ^{+}21 = h$ $(^{-}35 = h + ^{+}21)$

f. $^{-}18 - ^{-}13 = g$ $(^{-}18 = g + ^{-}13)$

g. $^{+}19 - ^{+}35 = r$ $(^{+}19 = r + ^{+}35)$

h. $^{+}45 - ^{-}17 = a$ $(^{+}45 = a + ^{-}17)$

i. $^{-}45 - ^{+}8 = d$ $(^{-}45 = d + ^{+}8)$

j. $^{-}12 - ^{+}16 = e$ $(^{-}12 = e + ^{+}16)$

2. Rewrite the following subtraction sentences as addition sentences. (different sentences may be used)

a. $^{-}4 - ^{-}2 = g$ $(^{-}4 = g + ^{-}2)$

b. $^{+}7 - ^{-}5 = i$ $(^{+}7 = i + ^{-}5)$

c. $^{-}3 - ^{-}7 = r$ $(r + ^{-}7 = ^{-}3)$

d. $^{+}2 - ^{-}8 = \ell$ $(^{+}2 = \ell + ^{-}8)$

e. $^{-}6 - ^{-}13 = c$ $(c + ^{-}13 = ^{-}6)$

f. $^{+}7 - ^{+}9 = a$ $(^{+}7 = a + ^{+}9)$

g. $^{+}12 - ^{+}5 = t$ $(^{+}12 = t + ^{+}5)$

h. $^{-}9 - ^{-}1 = s$ $(^{-}9 = s + ^{-}1)$

3. Choose an integer from Set A to use as an unknown addend in each of these.

a. $^{+}3 - o = ^{+}2$ $(^{+}1)$

b. $^{-}6 - a = ^{+}2$ $(^{-}8)$

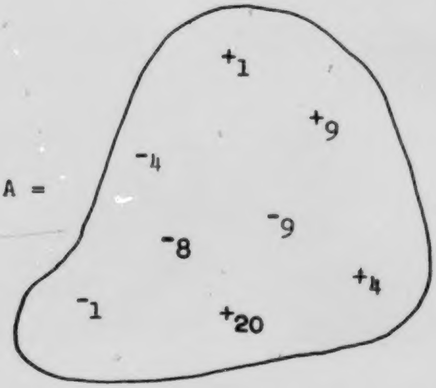
c. $^{+}16 - n = ^{+}7$ $(^{+}9)$

d. $^{+}12 - d = ^{-}8$ $(^{+}20)$

e. $^{-}10 - y = ^{-}9$ $(^{-}1)$

f. $^{+}5 - z = ^{+}14$ $(^{-}9)$

A =



4. **Rewrite as addition sentences. Then find the unknown addend.**

a. $^-17 - {}^+5 = n$
$({}^-17 = n + {}^+5,\ n = {}^-22)$

b. $^+10 - {}^-2 = e$
${}^+10 = e + {}^-2,\ e = {}^+12)$

c. $^-3 - {}^+8 = a$
$({}^-3 = a + {}^+8,\ a = {}^-11)$

d. $^+2 - {}^-6 = c$
$({}^+2 = c + {}^-6,\ c = {}^+8)$

e. $^+9 - {}^+3 = h$
$({}^+9 = h + {}^+3,\ h = {}^+6)$

f. $^-7 - {}^-32 = e$
$({}^-7 = e + {}^-32,\ e = {}^+25)$

g. $^+2 - {}^+19 = r$
$({}^+2 = r + {}^+19,\ r = {}^-17)$

h. $^-43 - {}^-16 = s$
$({}^-43 = s + {}^-16,\ s = {}^-27)$

5. **Copy and complete the following sentences by writing the correct sign of operation.**

a. $^-5\ \underline{(-)}\ {}^-7 = {}^+2$

b. $^+1\ \underline{(+)}\ {}^-1 = 0$

c. $^+13\ \underline{(-)}\ {}^-10 = {}^+23$

d. $^-9\ \underline{(+)}\ {}^+7 = {}^-2$

e. $^-1\ \underline{(+)}\ {}^+6 = {}^+5$

f. $^+18\ \underline{(-)}\ {}^-3 = {}^+21$

g. $^-2\ \underline{(-)}\ {}^-4 = {}^+2$

h. $^+26\ \underline{(-)}\ {}^+11 = {}^+15$

6. **Mark true or false.**

a. $^-2 + {}^-3 = {}^-3 + {}^-2$ (T)

b. $^+6 + {}^-4 = {}^-4 + {}^+6$ (T)

c. $^-5 - {}^-3 = {}^-3 - {}^-5$ (F)

d. $^-7 + {}^-5 = {}^-5 + {}^-7$ (T)

e. $^+3 - {}^-12 = {}^-12 - {}^+3$ (F)

f. $^-2 + {}^+3 = {}^+3 - {}^-2$ (F)

g. $^+16 + {}^-18 = {}^-18 + {}^+16$ (T)

h. $^+14 - {}^+9 = {}^+9 - {}^+14$ (F)

SUMMARY

Objective: To summarize the meaning, need, and use of the set of integers in addition and subtraction

To extend the commutative property of addition to the set of integers

To extend the closure property to the set of integers

To urge more able students to discontinue use of arrow diagrams when adding and subtracting integers

Materials: Dittoed copies of addition and subtraction charts on pages 241 and 243.

Suggested Teaching Procedure:

Exercise Set 16 is intended to summarize the ideas developed regarding integers. It may be used as independent work for pupils, as class discussion, or as a combination of the two.

These exercises are extremely important because they help children restate the commutative property of addition and realize that in this new set of numbers, the integers, subtraction as well as addition, is always possible.

The more able pupils should be urged to use the arrow diagrams only when necessary or to check their work; the less secure children, however, should be encouraged to use these aids. **Indeed, many children at this grade level may never advance beyond this point.**

* * *

There is a set of supplementary exercises included on pages 404-408 for use in review or as test items.

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REVIEW

Exercise Set 16

1. Add these integers. Try to add them without the use of arrow diagrams.

   a. ${}^{+}7 + {}^{+}3$ (${}^{+}10$)
   b. ${}^{-}6 + {}^{+}9$ (${}^{+}3$)
   c. ${}^{+}10 + {}^{-}5$ (${}^{+}5$)
   d. ${}^{-}4 + {}^{+}4$ (0)
   e. ${}^{-}14 + {}^{+}6$ (${}^{-}8$)
   f. ${}^{-}9 + {}^{-}19$ (${}^{-}28$)

2. Try to find the unknown addend without the use of arrow diagrams.

   a. ${}^{-}7 - {}^{-}3$ (${}^{-}4$)
   b. ${}^{+}6 - {}^{+}8$ (${}^{-}2$)
   c. ${}^{+}6 - {}^{+}3$ (${}^{+}3$)
   d. ${}^{-}12 - {}^{+}5$ (${}^{-}17$)
   e. ${}^{+}18 - {}^{-}2$ (${}^{+}20$)
   f. ${}^{+}5 - {}^{-}5$ (${}^{+}10$)

3. Perform these operations. Try to perform them without arrow diagrams.

   a. ${}^{+}3 + {}^{-}6$ ($= {}^{-}3$)
   b. ${}^{+}5 - {}^{-}7$ ($= {}^{+}12$)
   c. ${}^{-}8 + {}^{+}13$ ($= {}^{+}5$)
   d. ${}^{-}11 + {}^{-}4$ ($= {}^{-}15$)
   e. ${}^{-}4 - {}^{-}3$ ($= {}^{-}1$)
   f. ${}^{+}8 - {}^{+}10$ ($= {}^{-}2$)

4. BRAINTWISTER. Perform the following without the use of arrow diagrams.

   a. ${}^{-}625 - {}^{+}25$ ($= {}^{-}650$)
   b. ${}^{+}999 - {}^{+}1$ ($= {}^{+}998$)
   c. ${}^{-}455 + {}^{-}55$ ($= {}^{-}510$)
   d. ${}^{+}2{,}300 + {}^{-}300$ ($= {}^{+}2{,}000$)
   e. ${}^{-}7{,}225 + {}^{+}125$ ($= {}^{-}7{,}100$)
   f. ${}^{-}4{,}376 - {}^{-}4{,}376$ ($= 0$)

5. On any number line, how many units apart are:

   a. the 9 dot and the 4 dot? (5)

   b. the $^{-}6$ dot and the $^{+}3$ dot? (9)

   c. the $^{-}10$ dot and the $^{+}10$ dot? (20)

6. John has a score of $^{-}8$ points ("8 points in the hole") in a game. How many points would he need to earn to get to a score of $^{+}5$ points ("5 points out of the hole")? We can think of this in this way: "What integer must be added to $^{-}8$ to get a sum of $^{+}5$?"

$$^{-}8 + n = {}^{+}5$$
$$n = {}^{+}13$$

He would need to earn $^{+}13$ points.

   a. How many points would he need to earn to get to a score of $^{+}8$? ($^{+}16$)

   b. of $^{+}10$? ($^{+}18$)

   c. of $^{+}2$? ($^{+}10$)

   d. of 0? ($^{+}8$)

7. The lowest temperature ever recorded in the United States was 70° below zero at Roger's Pass, Montana. The highest temperature ever recorded in the United States was 134° at Death Valley, California. How many degrees higher was the temperature recorded at Death Valley than the temperature recorded at Roger's Pass? ($^{+}134 - 70 = {}^{+}204$ The temperature recorded at Death Valley was 204 degrees higher.)

8. Mt. Everest is 29,028 feet above sea level. The Dead Sea is 1,280 feet below sea level. How much higher is Mt. Everest than the Dead Sea? ($^{+}29{,}028 - {}^{-}1280 = {}^{+}30{,}308$. Mt. Everest is 30,308 feet higher than the Dead Sea.)

9. Complete the addition chart below. Add to each integer given in the left column the integer given in the top row. Use arrow diagrams if you need them.

ADDITION CHART

Addend

Addend +	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$
$^{+}4$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$	$^{+}6$	$^{+}7$	$^{+}8$
$^{+}3$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$	$^{+}6$	$^{+}7$
$^{+}2$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$	$^{+}6$
$^{+}1$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$
0	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$
$^{-}1$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$
$^{-}2$	$^{-}6$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$
$^{-}3$	$^{-}7$	$^{-}6$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$
$^{-}4$	$^{-}8$	$^{-}7$	$^{-}6$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0

10. Examine the chart and list as many relationships as you can.

(Answers will vary. They may include: Many sums are the same and these sums are in a diagonal line; each integer in a row is one greater than the integer to the left of it; each integer in a column is one less than the integer just above it; zero added to any integer is that integer. Also $^{+}3 + ^{-}2$ is the same as $^{-}2 + ^{+}3$ and other sums to show the commutative property for addition)

11. Complete the subtraction chart by filling in the unknown addends. Subtract from each integer given in the left column (the sum column), the integer given in the top row (the known addend row).

There are three ways to find integers to complete this chart. Suppose you are subtracting $^{-}3$ from $^{-}5$. You could write $^{-}5 - {}^{-}3 = n$. You can write this as an addition sentence $^{-}5 = n + {}^{-}3$.

A. Then use the addition chart. $^{-}5$ is the sum and $^{-}3$ is one addend. Find $^{-}3$ in the left column of the chart. $^{-}5$ is in the row to the right of $^{-}3$. The integer in the top row in this column is $^{-}2$. So $^{-}2$ is the number that is represented by n. $^{-}2$ belongs in the subtraction chart in the row to the right of $^{-}5$ and in the column headed by $^{-}3$.

B. You can use the counting method to find an unknown addend. You would count from the known addend ($^{-}3$) to the sum ($^{-}5$). This count would be to the left for 2 spaces, so n is $^{-}2$.

C. You can draw an arrow diagram to find the unknown addend. You would draw the arrow for the known addend ($^{-}3$) and for the sum ($^{-}5$). The arrow for the unknown addend would start at $^{-}3$ and have its head at $^{-}5$. The arrow would be labeled $^{-}2$.

SUBTRACTION CHART

Known Addend

Sum	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$
$^{+}4$	$^{+}8$	$^{+}7$	$^{+}6$	$^{+}5$	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$	0
$^{+}3$	$^{+}7$	$^{+}6$	$^{+}5$	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$	0	$^{-}1$
$^{+}2$	$^{+}6$	$^{+}5$	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$
$^{+}1$	$^{+}5$	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$	$^{-}3$
0	$^{+}4$	$^{+}3$	$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$
$^{-}1$	$^{+}3$	$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$
$^{-}2$	$^{+}2$	$^{+}1$	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$
$^{-}3$	$^{+}1$	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$
$^{-}4$	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$	$^{-}8$

12. **Examine the chart and list some relationships you noticed as you made the chart.**

> When two whole numbers are added, the order of the addends may be changed without changing the sum.

13. What property of addition is stated above? (Commutative property for addition)

14. Select at least five pairs of whole numbers. Add them to illustrate this property.

15. Select some pairs of integers and add them. Decide if this property also applies to the addition of integers. (It does)

16. If this property of addition of whole numbers also applies to integers, write the statement of the property on your paper. (When 2 integers are added, the order of the addends may be changed without changing the sum)

17. Fill in the chart below:

	Sum	Addend	Addend	Operation
1.	($^{-}3$)	$^{+}2$	$^{-}5$	(addition)
2.	($^{+}3$)	$^{-}3$	$^{+}6$	(addition)
3.	$^{+}7$	$^{+}9$	($^{-}2$)	(subtraction)
4.	(−4)	$^{-}6$	$^{+}2$	(addition)
5.	($^{-}6$)	$^{+}4$	$^{-}10$	(addition)
6.	$^{-}7$	$^{+}1$	($^{-}8$)	(subtraction)
7.	(0)	$^{+}8$	$^{-}8$	(addition)
8.	(−16)	$^{-}9$	$^{-}7$	(addition)
9.	($^{-}5$)	$^{+}7$	$^{-}12$	(addition)
10.	$^{+}3$	(+9)	$^{-}6$	(subtraction)
11.	(+15)	$^{+}12$	$^{+}3$	(addition)
12.	$^{-}11$	(−7)	$^{-}4$	(subtraction)
13.	(+36)	$^{+}24$	$^{+}12$	(addition)
14.	$^{+}18$	$^{-}2$	($^{+}20$)	(subtraction)
15.	(0)	$^{-}4$	$^{+}4$	(addition)

> The following exercises are included for the teacher to use as review or as suggestions for test items.

SUPPLEMENTARY EXERCISES

1. Mark true or false:

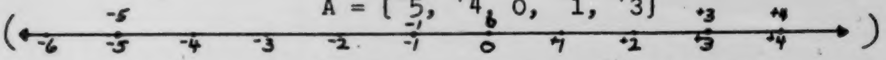
a.	$^{+}7 > 0$ (T)	e.	$^{-}2 > {}^{-}17$ (T)
b.	$^{-}3 > {}^{+}3$ (F)	f.	$^{-}6 > 0$ (F)
c.	$^{+}9 > {}^{-}9$ (T)	g.	$0 < {}^{+}4$ (T)
d.	$0 < {}^{-}6$ (F)	h.	$^{+}35 > {}^{-}56$ (T)

2. Write in words:

a. $^{+}7$ (positive seven) b. $^{-}3$ (negative three)

3. Draw a number line. Locate the members of Set A on it.

$A = \{{}^{-}5, {}^{+}4, 0, {}^{-}1, {}^{+}3\}$



4. Let Set C represent the set of integers. Write 3 subsets of Set C so that the union of these 3 subsets will include every integer. (Set of positive integers. Set of negative integers. Zero.) (This is the best answer. Others are possible.)

5. Write the integer that is

a.	3 less than zero. (-3)	d.	the opposite of $^{+}2$. (-2)
b.	4 more than $^{-}8$. (-4)	e.	7 less than $^{+}15$. (+8)
c.	5 more than $^{+}6$. (+11)	f.	9 more than $^{-}4$. (+5)

6. Copy and write the opposite of each integer below. Underline the greater integer of each pair:

a. $^{-}7$ (+7) (—) b. $^{+}6$ (—) (-6) c. 0 (0)

7. How many integers are there? (More than can be counted.)

8. Choose an integer from Column B to rename each number in Column A.

A	B
a. $^{-}3 + {}^{-}8 =$ (c)	a. $^{+}19$
b. $^{+}7 + {}^{-}9 =$ (f)	b. $^{-}31$
c. $^{+}14 + {}^{+}5 =$ (a)	c. $^{-}11$
d. $^{+}2 + {}^{+}3 =$ (e)	d. $^{-}7$
e. $^{-}21 + {}^{-}10 =$ (b)	e. $^{+}5$
f. $^{-}16 + {}^{+}9 =$ (d)	f. $^{-}2$

9. Complete each expression with an integer.

a. 2° below zero	(-2) degrees
b. 5,280 feet above sea level	(+5,280) feet
c. $200 profit	(+200) dollars
d. 38° north of the equator	(+38) degrees
e. a gain of 6 pounds	(+6) pounds

10. Complete:

a. If an integer is greater than its opposite, the integer is (positive).

b. If an integer is the same as its opposite, the integer is (zero).

c. Every negative integer is (less) than any positive integer.

11. Choose the correct answer:

$^{-}7$ is another name for:

(a) $^{-}2 + ^{+}5$

(b) $^{+}8 + ^{-}1$

(c) $^{+}7 + 0$

(d) $\underline{^{-}4 + ^{-}3}$

$^{+}1$ is another name for:

(a) $^{-}1 - ^{+}2$

(b) $0 - ^{+}1$

(c) $^{-}5 - ^{-}4$

(d) $\underline{^{-}1 - ^{-}2}$

0 is another name for:

(a) $^{+}9 + ^{+}9$

(b) $\underline{^{-}4 + ^{+}4}$

(c) $^{-}7 + ^{-}7$

(d) $^{-}5 + 0$

$^{+}6$ is another name for:

(a) $^{+}4 + ^{-}2$

(b) $^{-}6 + 0$

(c) $\underline{^{-}9 + ^{+}15}$

(d) $^{-}4 + ^{-}2$

$^{+}5$ could be the sum of:

(a) 2 negative integers

(b) a negative number and zero

(c) a pair of opposites

(d) $\underline{\text{2 positive integers}}$

$^{-}4$ could be the sum of:

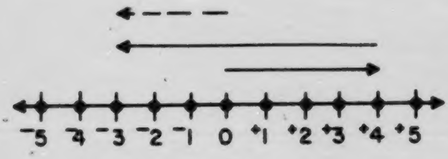
(a) 2 positive integers

(b) a positive integer and zero

(c) $\underline{\text{2 negative integers}}$

(d) a pair of opposites

12. Which mathematical sentence is diagramed?



a. $^{-}3 + ^{+}4 = ^{-}7$

b. $\underline{^{+}4 + ^{-}7 = ^{-}3}$

c. $^{-}7 + ^{-}3 = ^{+}4$

d. $^{-}3 + ^{+}4 = 0$

13. Mark true or false:

a. $^{+}2 + ^{+}4 = ^{+}4 + ^{+}2$ (T)

b. $^{+}2 + ^{-}4 = ^{+}4 + ^{-}2$ (F)

c. $^{-}2 + ^{-}4 = ^{+}4 + ^{+}2$ (F)

d. $^{-}2 - ^{-}4 = ^{-}4 - ^{-}2$ (F)

e. $^{+}7 - ^{-}5 = ^{-}5 - ^{+}7$ (F)

f. $^{+}7 + ^{+}5 = ^{+}7 - ^{-}5$ (T)

g. $^{-}7 + ^{-}5 = ^{-}5 + ^{+}7$ (F)

h. $^{+}7 - ^{+}5 = ^{-}7 - ^{+}5$ (F)

14. Add:

a. $^{+}9 + ^{-}3 =$ (+6)

b. $^{-}6 + ^{+}7 =$ (+1)

c. $^{+}5 + ^{+}4 =$ (+9)

d. $^{-}10 + ^{+}4 =$ (-6)

e. $^{-}11 + ^{+}7 =$ (-4)

f. $^{+}12 + ^{-}12 =$ (0)

15. Subtract:

a. $^{-}5 - ^{-}3 =$ (-2)

b. $^{+}6 - ^{-}2 =$ (+8)

c. $^{+}5 - ^{+}9 =$ (-4)

d. $^{+}2 - ^{-}4 =$ (+6)

e. $^{+}7 - ^{+}4 =$ (+3)

f. $^{-}7 - ^{-}11 =$ (+4)

16. Complete:

a. $^{+}6 - ^{-}3 = ^{-}3 +$ (+12)

b. $^{-}4 - ^{-}2 = ^{-}2 +$ (0)

c. $^{+}7 - ^{-}9 = ^{-}9 +$ (+25)

d. $^{-}4 - ^{-}7 = ^{-}7 +$ (+10)

e. $^{+}11 - ^{-}4 = ^{+}9 +$ (+6)

f. $^{+}3 - ^{+}17 = ^{-}6 +$ (-8)

g. $^{-}1 + ^{-}1 = ^{-}1 -$ (+1)

h. $^{+}3 - ^{+}3 = ^{-}3 +$ (+3)

17. Complete the table so that $a + b = ^{-}5$.

a	$^{-}6$	0	$^{+}5$	$^{-}9$	$^{+}8$	$^{-}15$	$^{+}3$	$^{+}12$	$^{-}13$
b	(+1)	(-5)	(-10)	(+4)	(-13)	(+10)	(-8)	(-17)	(+8)

18. Complete the table so that $a - b = {}^{-}1$.

a	$^{+}7$	$^{-}3$	$^{+}2$	$^{+}9$	$^{-}11$	$^{+}1$	0	$^{-}6$	$^{+}5$
b	(+8)	(-2)	(+3)	(+10)	(-10)	(+2)	(+1)	(-5)	(+6)

19. Mark true or false:

(T) a. Addition is always possible within the set of integers.

(T) b. Subtraction is always possible within the set of integers.

(T) c. Addition is always possible within the set of negative integers.

(F) d. Subtraction is always possible within the set of positive integers.

(F) e. Subtraction is always possible within the set of negative integers.

(T) f. The sum of a pair of opposites is zero.

(F) g. The sum of any integer and 0 is zero.

Chapter 5

COORDINATES

PURPOSE OF UNIT

The pupil's experience in mathematics has developed understanding of certain sets of numbers and operations with them. He has also been introduced to basic concepts about space and certain sets of points in space. The association of ideas about space with ideas about numbers has been limited to the number line and to use of numbers as measures of segments, angles, and regions.

This unit presents the idea that the location of sets of points in a plane can be described by the use of reference lines and numbers which are called coordinate systems. The particular coordinate system emphasized is the rectangular coordinate system, in which points are located with reference to two perpendicular lines.

MATHEMATICAL BACKGROUND

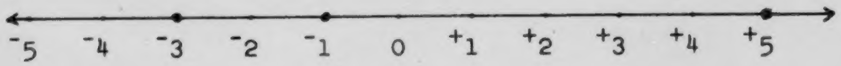
The idea of relating concepts of number and concepts of space is one which developed late in the history of mathematics. In fact, not until the work of René Descartes (1637) was the idea developed in a substantial fashion. Because of Descartes' contribution, the system of coordinates studied in this unit is sometimes called the Cartesian System.

The system is based on a one-to-one correspondence between a set of numbers and the set of points on a line. To set up a correspondence, we choose any point A on a line and agree to let it correspond to the number 0. We choose any other point B on the line and assign to it the number $^+1$. Using $\overline{AB}$ as unit, other points are located to correspond to the numbers $^+2$, $^+3$, $^+4$, and so on. In the opposite direction from A, points are located to correspond to $^-1$, $^-2$, $^-3$, and so on, so that the points corresponding to a number and its opposite are the same distance from the 0-point, A.

The positive or negative number which corresponds to a given point on the number line is called the coordinate of the point. The point on the number line which corresponds to a given number is called the graph of the number.

In this unit, the only numbers used as coordinates are the integers. However, other numbers will be used as coordinates in more advanced work. For example, there is a point on the line corresponding to each positive and negative rational number. Also there are still other points on the line corresponding to numbers called irrational numbers which cannot be expressed as rational numbers.

The process of graphing a set of numbers consists in locating on a number line the points which correspond to those numbers. Consider the set of integers: $\{{}^-3, {}^+5, {}^-1\}$ To graph this set, a line is chosen as a number line, the points corresponding to 0 and $^+1$ are selected and labeled, and a scale is marked off. A heavier, darker dot is then placed on the number line at each point which corresponds to a number in the set. The three heavier dots are the graph of this set. The segments connecting these dots (except for their endpoints) are not part of the graph. This is true because all points of the segments, other than the three marked by dots, correspond to numbers which are not members of the set to be graphed.



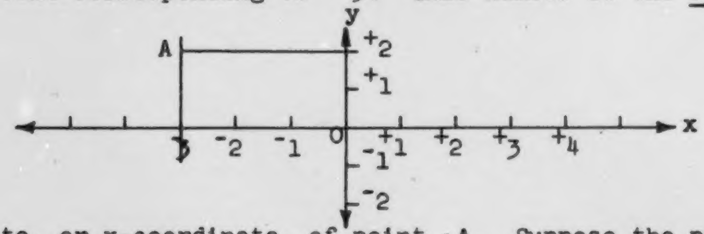
By using coordinates, many geometric questions may be answered by the methods of arithmetic. For example, suppose we wish to find the length of the segment whose endpoints are G, with coordinate $^-3$, and C, with coordinate $^+2$ on the number line above. If we subtract the coordinate of one point from the coordinate of the other point, we obtain an answer which tells us the number of units and direction from the first point to the second point. Since ${}^+2 - {}^-3 = {}^+5$, we know that point C is 5 units in a positive direction from point G. In a similar manner, ${}^-3 - {}^+2 = {}^-5$, shows point G is 5 units in a negative direction from point C. In either case, the points are 5 units apart and we say the length of $\overline{GC}$ (or $\overline{CG}$) is 5 units.

This system makes it possible to describe sets of points on a line by using coordinates. For example, we can speak of the segment whose endpoints have the coordinates ${}^-6$ and ${}^+7$, or the ray whose endpoint has coordinate ${}^+2$ and which contains the point whose coordinate is ${}^-1$. However, this system does not make it possible to describe geometric figures whose points are not all on the same line, such as a triangle. To do this, some way of using numbers to describe the location of any point in a plane is required.

A second number line is therefore introduced which intersects the first at the point corresponding to 0. In the coordinate system presented in this unit, the second number line is perpendicular to the first. The same point corresponds to 0 on both number lines, and the distance between the points corresponding to 0 and ${}^+1$ is the same on both lines. We now call each number line an axis. It is conventional to draw the first number line in a horizontal position. It is called the x-axis. Points corresponding to the positive numbers are to the right of the 0-point. The second number line, in a vertical position, is called the y-axis. Points above the 0-point correspond to positive numbers. The point of intersection (which corresponds to 0 on both axes) is called the origin.

To identify a point A in the plane, we assign it two coordinates, determined as follows: From A, lines are drawn perpendicular to the x-axis and the y-axis.

Suppose the perpendicular to the x-axis intersects the x-axis at the point corresponding to $^{-}3$. This number is the first

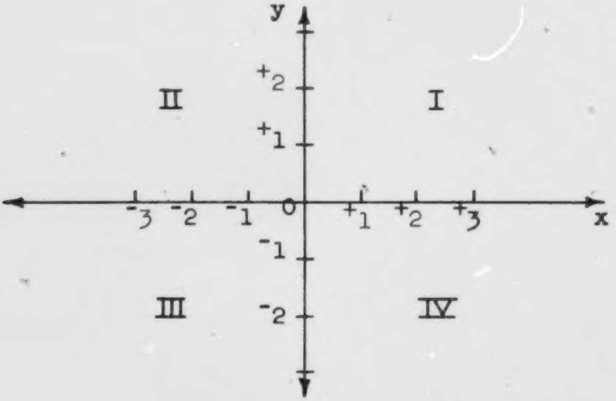


coordinate, or x-coordinate, of point A. Suppose the perpendicular to the y-axis intersects the y-axis at the point corresponding to $^{+}2$. This number is the second coordinate, or y-coordinate, of point A. The location of A is thus described by the pair of numbers, $^{-}3$ and $^{+}2$. We use a standard mathematical notation for the two coordinates by writing them within parentheses, with the x-coordinate always written first. The coordinates of point A are $(^{-}3, ^{+}2)$. Since the order is important--the point with coordinates $(^{-}3, ^{+}2)$ is not the same as the point with coordinates $(^{+}2, ^{-}3)$--these pairs of numbers are called ordered pairs.

The fact that $(^{-}3, ^{+}2)$ is the only ordered pair which can identify the point A with reference to these axes and the given unit is assured by these facts: from A there is only one line perpendicular to the x-axis; this line intersects the x-axis in exactly one point; and the point of intersection corresponds to exactly one number on the x-axis, $^{-}3$. Similarly, there is only one possible y-coordinate for A.

Suppose now we wish to graph the point P which corresponds to the ordered pair $(^{+}4, ^{-}5)$. The two coordinates uniquely determine the point P for the following reasons: At the point corresponding to $^{+}4$ on the x-axis, there is, in the plane, only one line perpendicular to the x-axis. Similarly, only one line in the plane is perpendicular to the y-axis at the point on the y-axis which corresponds to $^{-}5$. These two lines intersect in only one point, the point P, and its coordinates are $(^{+}4, ^{-}5)$.

The two coordinate axes separate the plane into five sets of points: the set of points of the coordinate axes themselves and the remaining four sets of the plane, called *quadrants*. These quadrants are numbered in a counter-clockwise direction (using Roman numerals) as in the sketch. We can thus give some information about a point and about its coordinates, by stating the quadrant or axis in which it lies. For example, if a point is in Quadrant II, then its x-coordinate is negative and its y-coordinate is positive; if a point is in the y-axis, then its x-coordinate is zero.

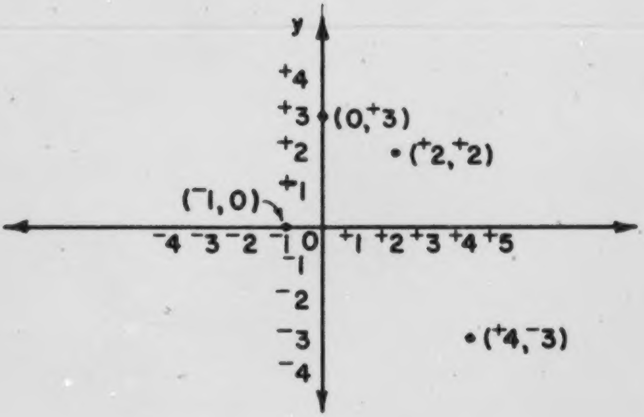


To assist in identifying accurately the points (or in representing accurately the points whose coordinates are known) it is customary to use graph paper on which equally spaced horizontal and vertical lines are printed. Any pair of perpendicular lines may be chosen as the x and y axes.

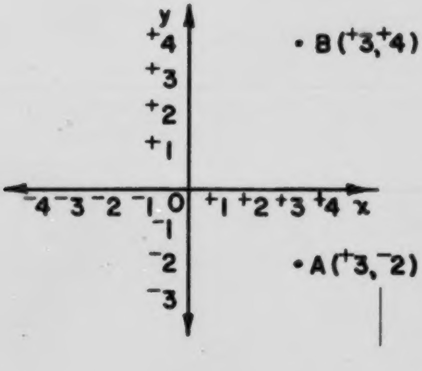
We have seen earlier that the graph of a single number is the point on a number line to which it corresponds. Similarly, the graph of an ordered pair consists of the point in the plane to which it corresponds. The figure on the next page shows the graph of this set of ordered pairs:

$$\{(^+4, {}^-3),\ (^+2, {}^+2),\ (^-1, 0),\ (0, {}^+3)\}$$

It is customary to label the graph of an ordered pair with its coordinates.



The method discussed earlier for finding the distance between two points of a number line may be used to find the distance beeween two points on any line parallel to the x-axis or parallel to the y-axis. Consider the points A($^+3$, $^-2$) and B($^+3$, $^+4$). Since these two points have the same x-coordinate, $\overleftrightarrow{AB}$ is parallel to the y-axis and 3 units to the right of it. Thus $\overline{AB}$ has the same length as the segment on the y-axis whose endpoints have coordinates $^+4$ and $^-2$, and this length, as we have seen, can be found by finding the difference of these y-coordinates. Since $^+4 - {}^-2 = {}^+6$, point B is 6 units above A. Likewise, $^-2 - {}^+4 = {}^-6$, so A is 6 units below B. Thus the length of $\overline{AB}$ is 6 units.

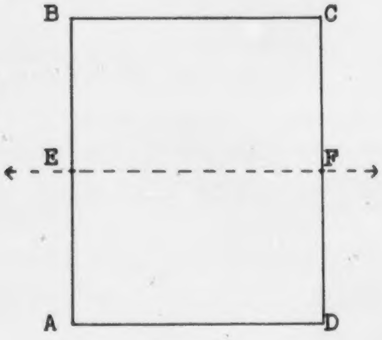


It is interesting to explore the effect on a polygon of adding an integer n to the x-coordinate of each vertex, leaving the y-coordinate unchanged. We find that this operation "moves" the polygon n units to the right if n is positive and n units to the left if n is negative, but the new polygon is congruent to the original one. In a similar manner, if an integer m is added to the y-coordinate of each vertex, the polygon is moved upward by m units if m is positive and downward by m units if m is negative.

It is also interesting to explore the effect of multiplying both coordinates of each vertex of a polygon by the same number. We find that the polygon whose vertices have the new set of coordinates will have the same shape, but not the same size, as the original polygon. (The polygons are said to be similar.)

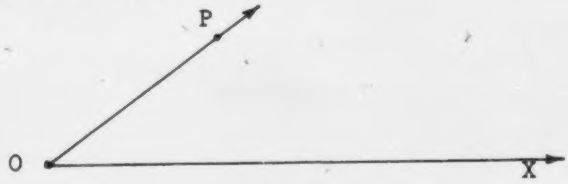
Coordinates may also be employed in the study of reflections. The concept of congruent figures is a familiar one, as is the informal test for congruence by placing a tracing of one figure on the other, to see whether each point of one falls on a point of the other. In a similar fashion, one figure is the "reflection" of another if they are so placed in the plane that by "folding" along a line in the plane the congruence of one figure to the other can be demonstrated. One figure is thus seen to be the "mirror image" of the other. The particular lines of "reflection" emphasized in this unit are the coordinate axes. If figure A is a reflection of figure B with respect to the y-axis, then for every point of A there will be a point of B with the same y-coordinate but with the x-coordinate replaced by its opposite; and likewise for every point of B there will be a corresponding point of A. If figure A is the reflection of figure B with respect to the x-axis, then for every point of A there will be a point of B with the same x-coordinate but with the y-coordinate replaced by its opposite.

In some cases half of a figure is the reflection of the other half with respect to some line. Figures of this type are said to be *symmetric* and the line is called the *line of symmetry* or *axis of symmetry*. This means that a symmetric figure is a very special case of a reflection--that reflection in which the line of reflection goes "right through the middle" of the figure.



Polygon ABCD is a symmetric figure. The reflection of $\overline{EA}$, $\overline{AD}$, and $\overline{DF}$ in $\overleftrightarrow{EF}$ are $\overline{EB}$, $\overline{BC}$, and $\overline{CF}$, respectively. Also, the reflections of $\overline{EB}$, $\overline{BC}$, and $\overline{CF}$ in $\overleftrightarrow{EF}$ are $\overline{EA}$, $\overline{AD}$, and $\overline{DF}$, respectively. The line of reflection, $\overleftrightarrow{EF}$, is also called the line (or axis) of symmetry of the symmetric figure ABCD.

A coordinate system different from the system of rectangular coordinates discussed above is the system of polar coordinates. In this system, a horizontal ray OX extending to the right of the endpoint O is used for reference. The two numbers used to describe the position of a point in the plane are the measure of a segment and the measure of an angle. The endpoint of the ray, O, is called the *pole*. In the sketch below, the position of point P may be described by stating the measure of angle POX and the measure of $\overline{OP}$.



It is understood that the measure of the angle is to be made "counterclockwise"--that is, OP is in the direction from OX which is the opposite from that suggested by the movement of the hands of a clock. It is clear that if $\overline{OP}$ has a length of 6 units and $m\angle POX$ in degrees is 40, point P is uniquely determined.

The polar coordinate system is not discussed in this unit, but one of the methods for locating a point discussed informally in the first section makes use of the idea of this coordinate system.

OVERVIEW OF THE UNIT

The first section of the unit presents the problem of finding a way to identify a point in a plane. Three methods are discussed informally. One of these depends upon the ideas of a rectangular coordinate system (the one we will use), and another of a polar coordinate system, although these terms are not used. The third depends upon the fact, familiar to the pupil, that any triangle whose sides are segments with three given lengths is congruent to any other triangle whose sides have the same lengths.

Coordinate systems are then introduced, first on a line and then in a plane. *The coordinate of a point on a line is defined as the number which corresponds to it, and the graph of a number as the point which corresponds to it.* A method for finding the length of a segment by means of coordinates is developed. The use of two perpendicular number lines and an ordered pair of numbers to describe the position of a point in the plane is explained, and practice is provided in finding coordinates of points and graphing ordered pairs. In some exercises, segments connecting pairs of points to form polygons are used, and the effect of changing coordinates on the size and shape and position of the polygon is explored.

The final sections deal with the use of coordinates in studying the reflection of a set of points in the x-axis and in the y-axis, and with the concept of symmetric figures.

TEACHING THE UNIT

In this unit each section is divided into two parts--Exploration and Exercises. The Exploration is intended to be a teacher-directed class activity. During the exploration period the pupils should participate fully. In most cases the pupils' books will be closed. However, for some sections the teacher may want the pupils to work through the Exploration item by item with their books open. After the understanding is developed, or the skill is learned, or the generalization is realized, the pupils may open their books.

The Exploration is included in the pupil book and serves as a printed record of the activity in which the class has participated. The teacher should decide the extent to which the Exploration should be repeated with the books open. In most cases, the pupils will glance quickly over the printed Exploration and then proceed to the Exercises. However, if the pupils have had trouble with the section, the teacher may prefer to have the pupils read the Exploration in detail to reinforce the ideas that have been developed. This may be done as a group activity or an individual activity.

Independent work by the pupils should be done on the Exercises. The Exercises will help provide further understanding or will provide for drill and retention of material. The Exercises also develop concepts and therefore should be used as a basis for class discussion after pupils have had opportunity for individual study.

DESCRIBING LOCATIONS

Objective: To introduce the general problem of describing a location and to develop the notion that there are several ways of describing a location.

Vocabulary:

Materials Needed:

Teacher: chalkboard compass, blackboard protractor, straightedge.

Pupil: pencil, paper, protractor, compass, straightedge.

Teaching Procedure:

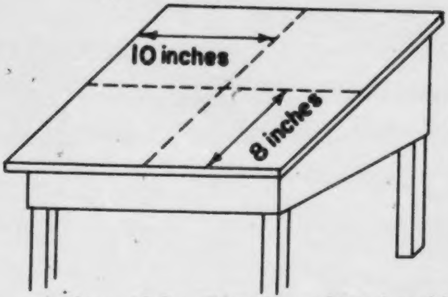
Follow the Exploration in the pupil text. Give the children opportunity to "discover " and to discuss.

Essentially there are three methods presented for locating a point. The methods are by using rectangular coordinates, polar coordinates, and triangulation.

Encourage the pupils to proceed step by step along with the Exploration in order that they may understand and "get the feel" of what is being done. Solicit ideas from the children about other ways of locating a point.

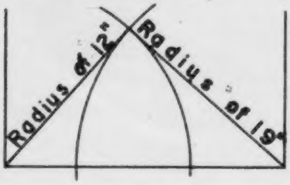
It is often helpful for the pupils to see an example of each method of locating a point before they try to do it. Place a dot as a representation of a point on the chalkboard and find its location using the three methods proposed in the exploration.

Martha's method illustrates the use of rectangular coordinates--the method used in this unit. A sketch might look like this:

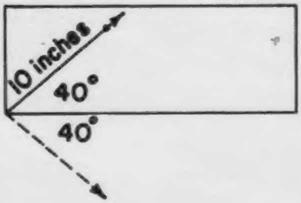


After gaining this first understanding relative to locating a point of a plane the children should be prepared to work the exercises and arrive at the desired conclusions. Two other methods are presented for the purpose of contrast; however, after the exploration only "Martha's method" will be used.

Jane's method is one of triangulation. The point of intersection of the circles drawn from each corner would mark the location of the point. There would, of course, be another point of intersection of these two circles, but this point would not be on the desk.



Joe's method illustrates the use of polar coordinates. This system of locating positions is commonly used in piloting, for example. A sketch of Joe's method might be:



There is another ray that makes an angle whose measure in degrees is 40 but it would be in the other half plane, as shown by the dotted line in the sketch.

Chapter 5

COORDINATES

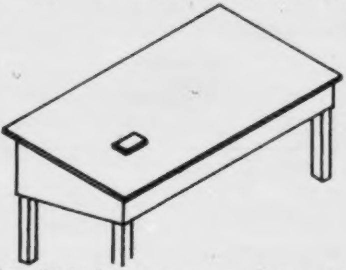
DESCRIBING LOCATIONS

Exploration

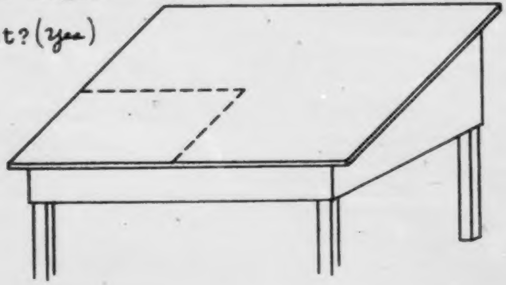
1. How many points are there in a plane? (More than can be counted) Is each point different from every other point? (yes, in location.) How can we find a way to identify a particular point? (answers will vary.)

2. Think of the top of your desk as a part of a plane. Place a small object to represent one point on your desk top. How could you describe its location? (Varying ways. These ways are described in Ex. 3, 4, 5, 6.)

3. Ted said, "The point is 7 inches from the lower left-hand corner of my desk." Does this give you enough information? (No) How many points are 7 inches from the corner of the desk? (Many)



4. Martha said, "I didn't measure from the corner. My point is 7 inches from the left-hand edge and 8 inches from the lower edge." Is Martha's information enough to locate the point? (yes)



5. Jane said, "I can use Ted's information and some more information to describe a point. My point is 7 inches from the lower left-hand corner and 19 inches from the lower right-hand corner." Does this give you enough information? (yes)

   How many points are 7 inches from the lower left-hand corner? (many) How many points are 19 inches from the lower right-hand corner? (many) How many points of the desk top are at the correct distance from both corners? (one. a triangle is formed; the method is by triangulation.)

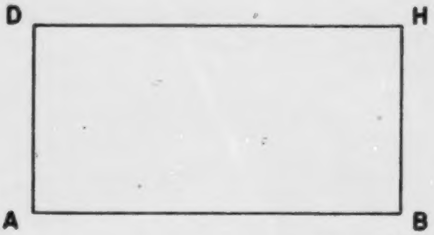
6. Joe said, "I can use Ted's information and some different information. My point is 7 inches from the lower left-hand corner. It is on a ray that makes a 40° angle with the ray from the corner on the lower edge of my desk." How many such rays are there? (one) Does Joe's method work? [yes, (10, 40°) are called polar coordinates.]

7. Are there at least three ways of describing the location of a point on the top of the desk? (yes) Can you think of others? (Answers will vary.)

Exercise Set 1

Suppose a rectangular region 6 inches long and 4 inches wide represents a picture, and a point C is a particular point of the picture.

1. Use your ruler and protractor to draw a rectangular region to represent the picture. Label it as shown below. $\overline{DH}$ is 6 inches in length. $\overline{AD}$ is 4 inches in length.



2. Suppose C is 5 inches from A and $3\frac{5}{8}$ inches from B.
   a) Use your compass to locate C. Is C exactly one point of the picture? (Yes)
   b) What property of triangles is illustrated? (Given the length of 3 sides of a triangle, the triangle is determined.)
   c) What information was used to locate C? (Distance of C from A and from B) What fixed points were used? (A and B) How far apart are these fixed points? ($\overline{AB} = 6$ in., $\overline{AC} = 5$ in., $\overline{BC}$ is about $3\frac{5}{8}$ in.)

NOTE: See T.C. page 425 for drawing.

3. a) Make another copy of the rectangular region.

   Draw $\overrightarrow{AC}$ so that the union of $\overrightarrow{AC}$ and $\overrightarrow{AB}$ is an angle of 37°. (For drawing see TC page 425.)

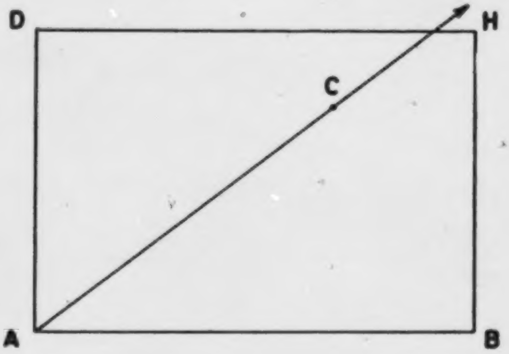
   b) On $\overrightarrow{AC}$ locate point C to make $\overline{AC}$ 5 inches long.

   c) Is C exactly one point of the picture? (Yes)

   d) What information was used to locate C? (The length of $\overline{AC}$ and the angle CAB of 37°.)

   e) What fixed point and line were used? (Point A and $\overleftrightarrow{AB}$)

   Your drawing should look about like this:



4. Copy the rectangular region again.

   a) Locate a point E on $\overline{AB}$ so that $\overline{AE}$ is 4 inches long.

   b) Draw a ray. Put its endpoint at E. Make it so that it and $\overrightarrow{EA}$ form a right angle.

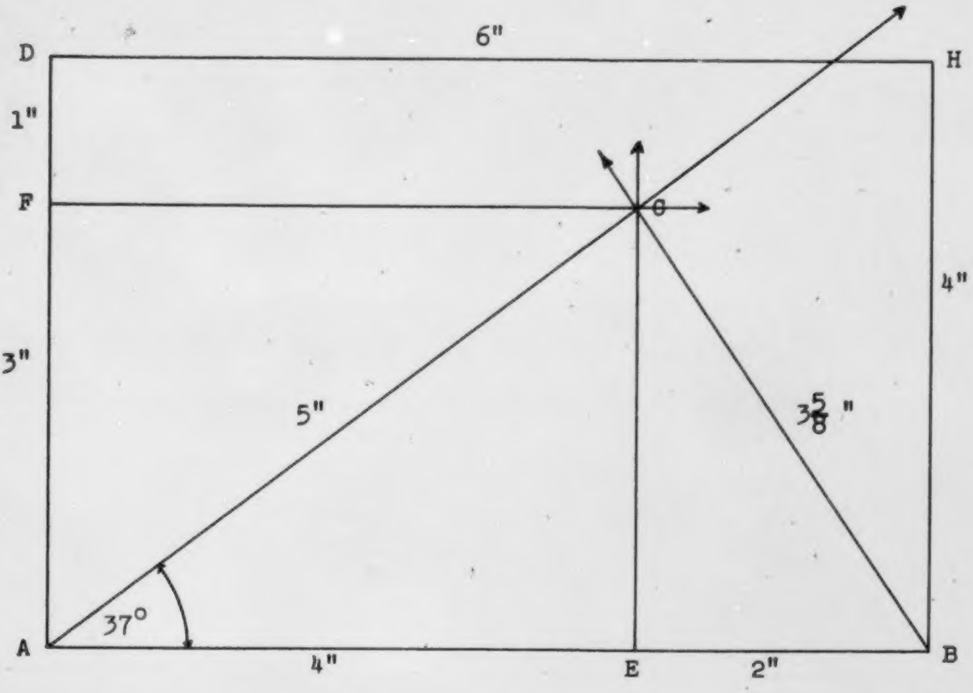
   c) Locate a point F on $\overline{AD}$ so that $\overline{AF}$ is 3 inches long.

   d) Draw in the rectangular region a ray with endpoint F perpendicular to $\overrightarrow{FA}$.

   e) Does the intersection of the rays you have drawn locate exactly one point C? (Yes)

Answers: Exercise Set 1

Exercises 1-2-3-4


D
6"
H
1"
F
G
4"
3"
5"
$3\frac{5}{8}$"
37°
A
4"
E
2"
B

f) What information was used to locate C? (The length of $\overline{AF}$ and $\overline{AE}$ and the measure of $\angle CFA$ and $\angle CEA$.)

g) What fixed lines were used? ($\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$)

5. Look at Exercise 2 in this Exercise Set. Is the method used in it the same as the method used by Jane in Exercise 4 of the Exploration? (yes.)

6. Look at Exercises 3 and 4 in this Exercise Set.

   a) Which of these exercises uses the same method that Joe used in Exercise 6 of the Exploration? (Ex.3)

   b) Which of these exercises uses the same method that Martha used in Exercise 4 of the Exploration? (Ex 4)

II. COORDINATES ON A LINE

Objective: To develop the concepts of coordinate of a point and graph of a number.

Vocabulary: coordinate, graph, horizontal, vertical

Materials: straightedge, pencil, paper

Teaching Procedure:

A short review of the general concepts of the number line would be most helpful here. The previous chapter on integers is a very good reference. The amount of review depends on the ability of the class to retain the facts learned and also the depth of learning which took place in the previous unit. After this review, or if no review is necessary, the teaching procedure should follow the exploration in the pupil's book.

Call particular attention to Exercise 4 of the Exploration as this is the definition of a coordinate and the "heart" of this section. Emphasize that a coordinate of a point on a line tells both distance and direction of a point from a given point.

Discuss at some length with the pupils the meaning of the word graph as used in this section. Begin to use the word "graph" rather than "plot the points" as soon as possible.

As in the chapter on the integers, $^{+}3$ is read "positive 3," and $^{-}3$ is read "negative 3."

COORDINATES ON A LINE

Exploration

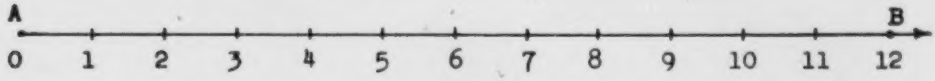
The methods you have considered for locating a point in a plane all involved using:

a) at least one fixed point and at least one line from which measurements were made; and

b) at least two measurements of segments or angles.

1. Think of a situation in which you are given a fixed point A on $\overleftrightarrow{AB}$. Can you describe the position of another point C by just one measurement? (yes) Where must C lie if this is possible? (to the right of A or to the left of A on $\overleftrightarrow{AB}$)

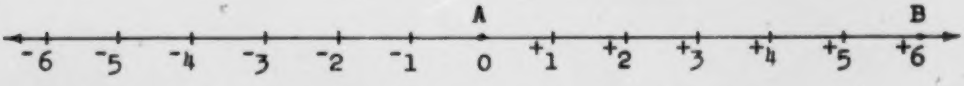
2. Look at the number ray below.



If you know the distance of C from A is 6 units and that C is on $\overrightarrow{AB}$, do you know exactly where C is? (yes, 6 units to the right of A.)

3. Now look at the number line below.

a) What kind of numbers are shown on this number line? (Negative and positive numbers)



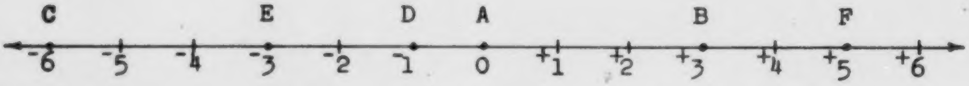
b) If you know the distance of a point C from A, and that C is on $\overleftrightarrow{AB}$, how many different points could be named C? (Two. Either to the right or left of A.)

c) What must you know beside the distance in order to locate exactly one point C ? (the distance of C from A)

4. What kind of numbers tell both the direction and distance of a point from A? (positive and negative numbers)

A number that tells both distance and direction of a point on a line from the 0-point is called the coordinate of the point.

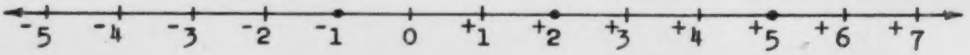
5. On the number line below, what is the coordinate of B? (+3) of C? (-6) of D? (-1)



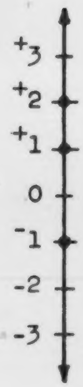
6. What point has the coordinate $^{-}3$? (E) What point has coordinate $^{+}5$? (F)

When you mark on a number line the points which have a certain set of numbers as coordinates we say you are drawing a graph of the set of numbers.

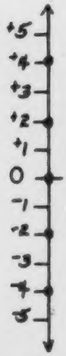
Below is shown the graph of a set of integers. The three heavy dots are the graph of the set $\{^{-}1, ^{+}2, ^{+}5\}$. Is this a different kind of graph from those you have seen before?
(The answer will probably be yes.)



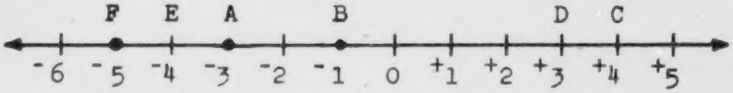
7. Sometimes vertical (up and down) lines are used. What set of integers is graphed on this line?
$\{^+2, ^+1, ^-1\}$



8. Draw a vertical number line and graph this set of integers:
$\{^-4, ^+4, ^-2, ^+2, 0\}$.



9. What is the coordinate of the point half-way between A and B on the number line below? $(^-2)$ Is the coordinate an integer? (yes)



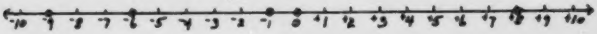
10. What number should be the coordinate of a point half-way between C and D? $(^+3\frac{1}{2}$ or $^+\frac{7}{2})$ half-way between E and F? $(^-4\frac{1}{2}$ or $^-\frac{9}{2})$ Are these coordinates integers? (No)

Many points have as coordinates numbers which are not integers. In this unit, however, we shall use points whose coordinates are integers.

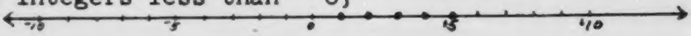
Exercise Set 2

Draw number lines showing the integers from $^-10$ to $^+10$ and graph these sets. Use some horizontal and some vertical number lines. (Note: Due to space for answers, only horizontal number lines were used here. Encourage the pupils to draw *some* vertical number lines.)

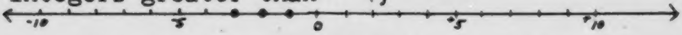
1. $\{^-9, ^-6, ^-1, 0, ^+8\}$


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
+6
+7
+8
+9
+10

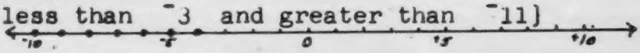
2. {The positive integers less than $^+6$}



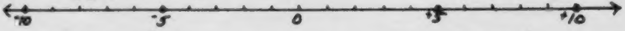
3. {The negative integers greater than $^-4$}



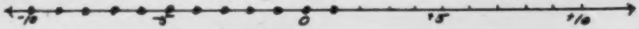
4. {The integers less than $^-3$ and greater than $^-11$}



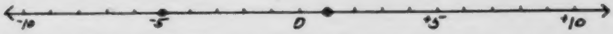
5. {The positive integers less than $^+11$ which are divisible by 5, and their opposites}



6. {The integers less than $^+2$ and greater than $^-11$}



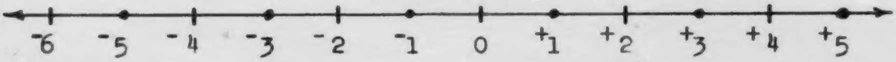
7. {The integer which is 3 greater than $^-2$ and the integer which is 3 less than $^-2$}



List or describe the sets of integers whose graphs are shown below.

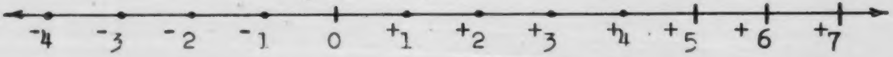
$\{^-5, ^-3, ^-1, ^+1, ^+3, ^+5\}$

8.


-6
-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5

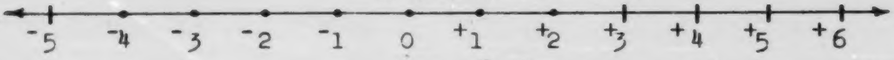
$\{^-4, ^-3, ^-2, ^-1, ^+1, ^+2, ^+3, ^+4\}$

9.


-4
-3
-2
-1
0
+1
+2
+3
+4
+5
+6
+7

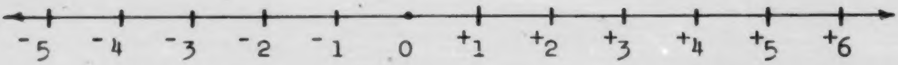
$\{^-4, ^-3, ^-2, ^-1, 0, ^+1, ^+2\}$

10.


-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
+6

$\{0\}$

11.


-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
+6

III. COORDINATES IN A PLANE

Objective: To develop the understanding of identifying a point by using two number lines and an ordered pair of numbers. To gain facility in identifying points using ordered pairs, to introduce the use of graph paper, and to use the terminology of x- and y-axes, x- and y- coordinates with reference to the point of origin.

Vocabulary: ordered pair, perpendicular, graph paper, x-axis, y-axis, axes, x-coordinate, y-coordinate, origin

Materials:

Teacher: straightedge (yardstick), chalk, square chalkboard or poster board

Pupil: pencils, graph paper, straightedge

Teaching Procedure:

This section should be divided into two lessons. It might be beneficial for some pupils to keep books open during these explorations. The first exploration and Exercise Set 3 will constitute a lesson with the second exploration and Exercise Set 4 making the next lesson.

Remind the children that a number line may extend in any direction but for our use here the two number lines we will consider are perpendicular lines, one being vertical and the other horizontal. It would be helpful to make a diagram using the chalkboard as the lesson proceeds through the first Exploration. It would be a big help if one of your chalkboards is ruled like graph paper. A music staff writer might be used to construct one. Perforated grid stencils using chalk dust from erasers may be constructed for blackboard use.

Bring out the fact that in an ordered pair, such as we are using, the first number tells the direction (left or right) and the distance from zero on the horizontal line. The second number of the pair gives the distance and direction (up or down) on the vertical line. The order is always the same, the number on the horizontal line first and then the number on the vertical line. This, of course, is just a convention, but it is a standard one.

The graphing of ordered pairs suggested in the Exploration Exercise 2 should be done as a class exercise in order to provide guidance to pupils. They will need help in determining which horizontal and vertical line to use for their problems. After work on numbering the lines, pupils will be more able to work Exercise Set 3 independently. Pupils should be required to label axes as described in the text, and to label points with their coordinates.

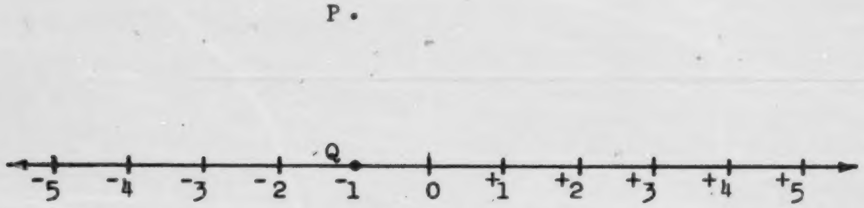
The second Exploration introduces the terms x-axis, y-axis, x-coordinate, y-coordinate, and origin. Now the pupils are ready to develop real skill in graphing. Exercise 3, might well be done as a demonstration at the chalkboard. Mark off a squared section of the chalkboard, draw the axes, and identify a few points. Some children might also identify points on the squared section at the board. After this, pupils should be ready to do Exercise Set 4 independently.

After the children have had practice in making and marking coordinate axes, much time can be saved by furnishing them duplicated sheets with axes and numerals marked. Numeral labels about $\frac{1}{2}$ inch or $\frac{3}{8}$ inch apart seem to be about right.

COORDINATES IN A PLANE

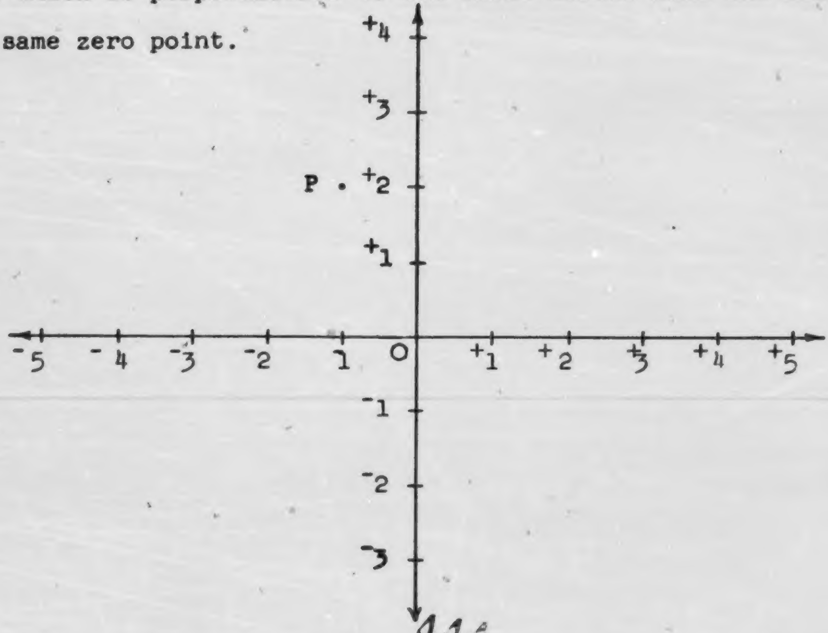
Exploration

Look at the number line and the points P and Q below.



Since Q is on the number line we can state its position by naming its coordinate, ⁻1. On the other hand, since P is not on this number line, we cannot state its position by naming its coordinate. It seems to be directly above ⁻1 on the line, but we need a way to tell how far above the line it is.

We can find a way to do this by using a second number line which is perpendicular to the first number line and has the same zero point.



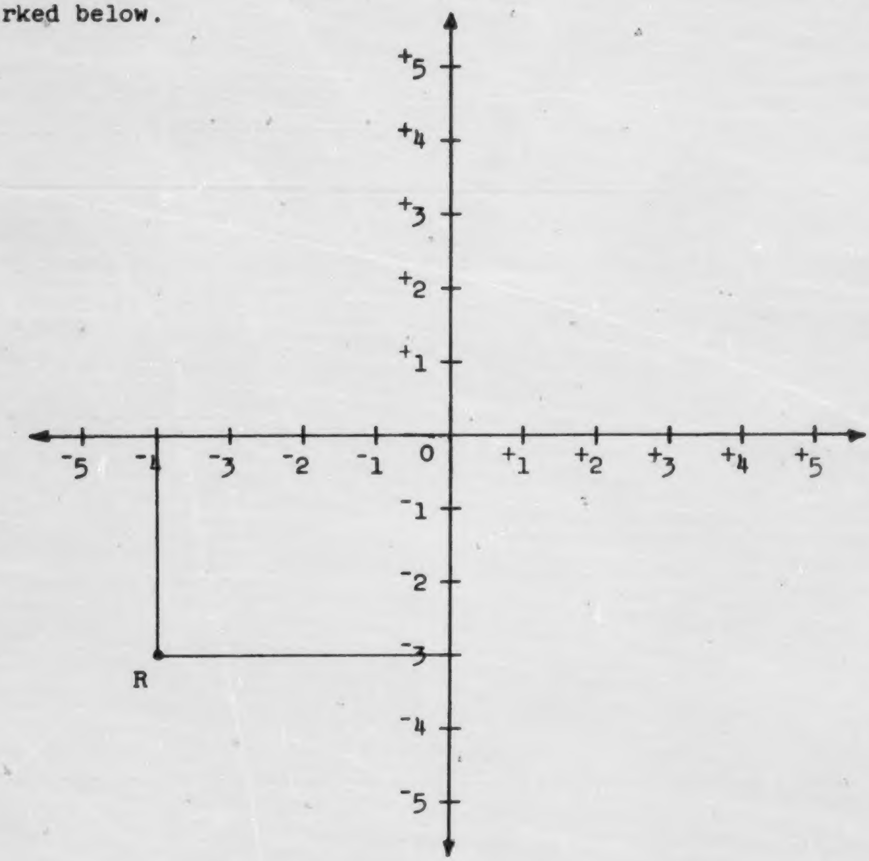
We see now that P is directly to the left of $^{+}2$ on the vertical number line. We can describe its position by using the two numbers, $^{-}1$ and $^{+}2$. Is there any point except P which is exactly above $^{-}1$ on the horizontal number line and also exactly to the left of $^{+}2$ on the vertical number line? (No)

What is meant by "exactly above" and "exactly to the left of" in the last two sentences? ("Exactly above" means on a line perpendicular to the horizontal line at the point $^{-}1$. "Exactly to the left of" means on a line perpendicular to the vertical line at the point $^{+}2$.)

The position of P can thus be described not by one number, but by the pair of numbers, ($^{-}1$, $^{+}2$). The numbers $^{-}1$ and $^{+}2$ are called coordinates of the point P.

The first number tells the number exactly below it (in this example) on the horizontal number line. The second number tells the number exactly to the right (in this example) of P on the vertical number line. The order in which the numbers are named is important, so ($^{-}1$, $^{+}2$) is called an ordered pair.

Let us think more precisely about finding the numbers which describe the position of a point. Look at point R marked below.

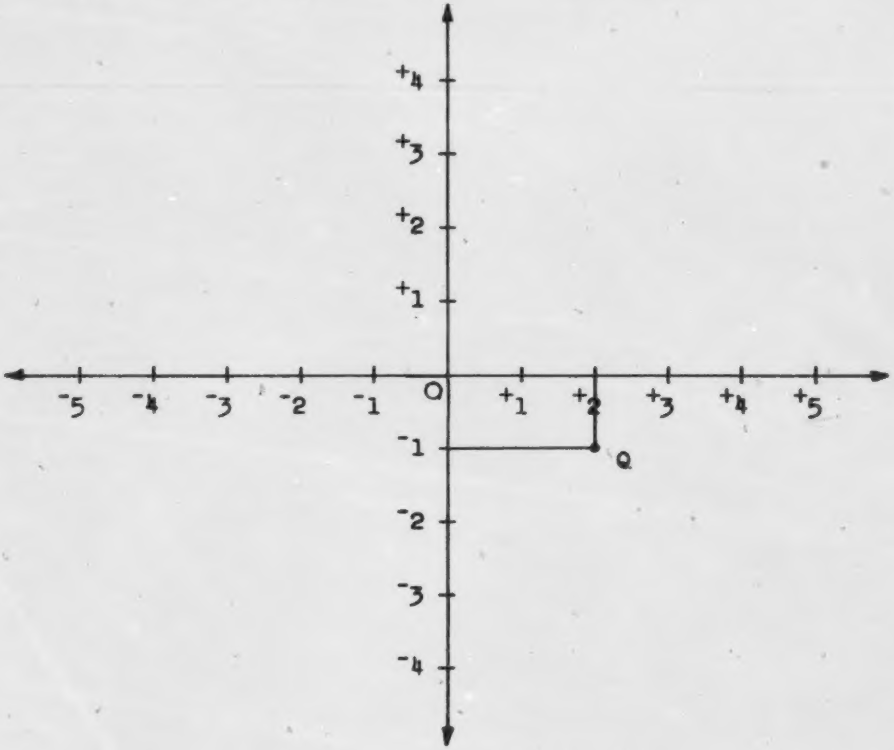


A line segment from R is drawn perpendicular to the horizontal number line. It intersects the number line at $^{-}4$.

A line segment from R is drawn perpendicular to the vertical number line. It intersects the number line at $^{-}3$. The location of R is described by the ordered number pair ($^{-}4$, $^{-}3$).

We can use perpendicular lines to find the position of a point.

Consider the point which is described by the ordered pair $(^{+}2, ^{-}1)$. Look at the drawing below.



Since the first number is $^{+}2$, find the point for $^{+}2$ on the horizontal number line. A line segment perpendicular to the number line is drawn at this point.

Find the point $^{-}1$ on the vertical number line.

A line segment perpendicular to the number line is drawn at this point.

The two perpendicular line segments intersect at the point labeled Q. Q is the point whose position is described by the ordered pair $(^{+}2, ^{-}1)$.

Its first coordinate is ? (+2).

Its second coordinate is ? (-1).

Do the ordered pairs $(^{+}2, ^{-}1)$ and $(^{-}1, ^{+}2)$ describe two different points in the plane? (yes)

Briefly, we can think:

To locate the point $(^{-}4, ^{-}3)$, start at (0, 0) count 4 units to the left and then 3 units down.

To locate $(^{-}1, ^{+}2)$, count 1 unit to the ? (left) and then 2 units ? (up).

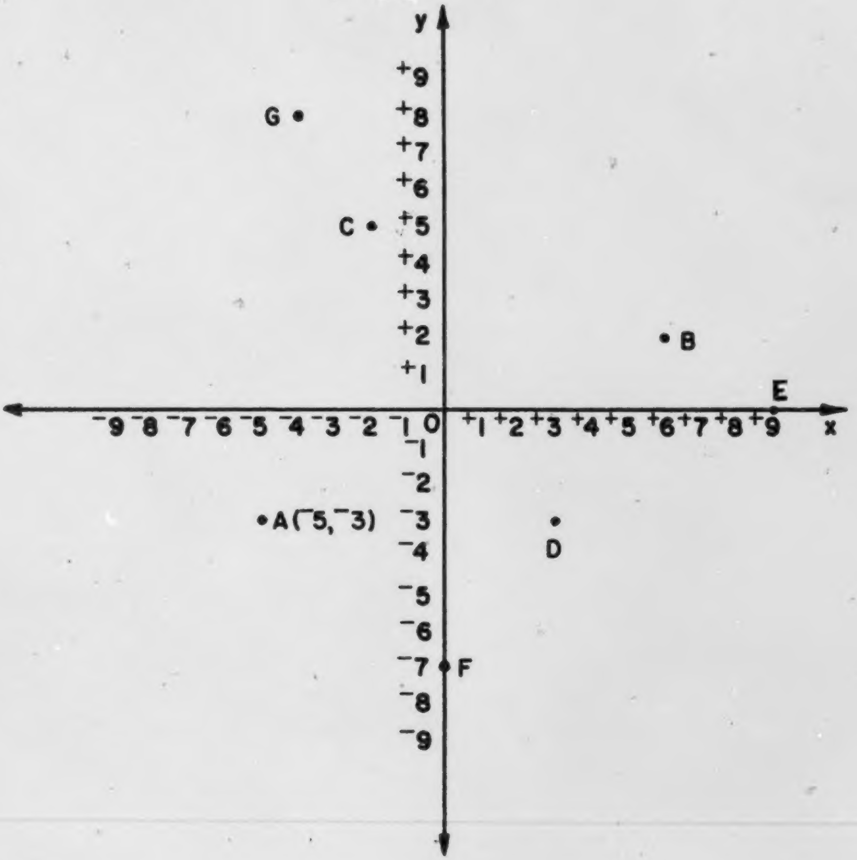
To locate $(^{+}2, ^{-}1)$, count ? (2) units to the ? (right) and then ? (1) unit ? (down).

To assist in describing accurately the position of points it is customary to use graph paper. On graph paper sets of perpendicular lines are printed forming segments of equal length. Any pair of perpendicular lines may be chosen as the number lines.

The ordered pair $(^-5, ^-3)$ is graphed below and the point it identifies is labeled A. Notice the ordered pair is written in parentheses beside the point.

1. Write the ordered pairs which are the coordinates of points B, C, D, E, F, and G. Write your answer like this: $A(^-5, ^-3)$

   $B(^+6, ^+2)$, $C(^-2, ^+5)$, $D(^+3, ^-3)$,
   $E(^+9, 0)$, $F(0, ^-7)$, $G(^-4, ^+8)$



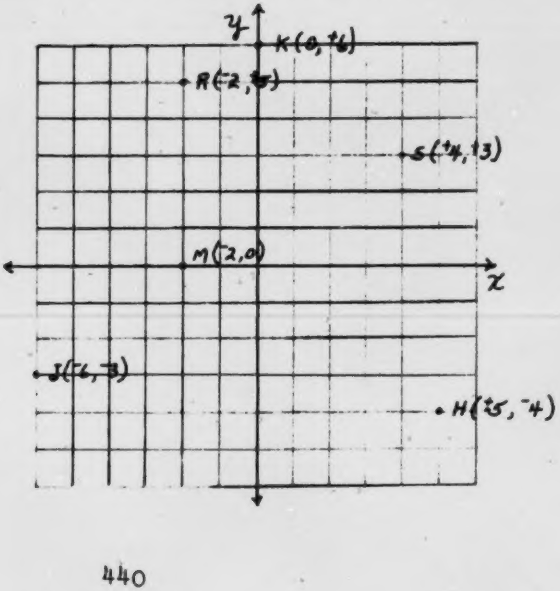
2. Use a sheet of graph paper with lines one-half inch apart. Choose two perpendicular lines for number lines and draw heavy lines on them to show the lines you have chosen. Label the number lines from $^{-}6$ to $^{+}6$.

Graph these ordered pairs. Label each with its letter and its coordinates.

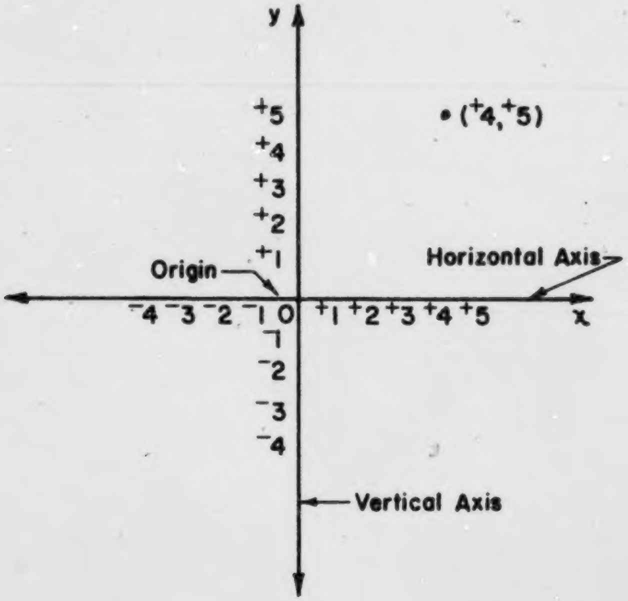
H($^{+}5$, $^{-}4$)
J($^{-}6$, $^{-}3$)
K(0, $^{+}6$)
M($^{-}2$, 0)
R($^{-}2$, $^{+}5$)
S($^{+}4$, $^{+}3$)

When number lines are used in this way, we call each number line an axis. The horizontal number line is called the x-axis and the vertical number line is called the y-axis. The point of intersection of the x-axis and the y-axis is called the origin.

(answer to 2 above)



When you draw graphs you should always label the axes ("axes" is the plural of axis) in a certain way as shown below.


y
+5
+4
+3
+2
+1
• (+4, +5)
Origin
Horizontal Axis
−4 −3 −2 −1 0 +1 +2 +3 +4 +5
x
−1
−2
−3
−4
Vertical Axis

Write "x" and "y" near the arrows on the rays of the axes which show the positive integers.

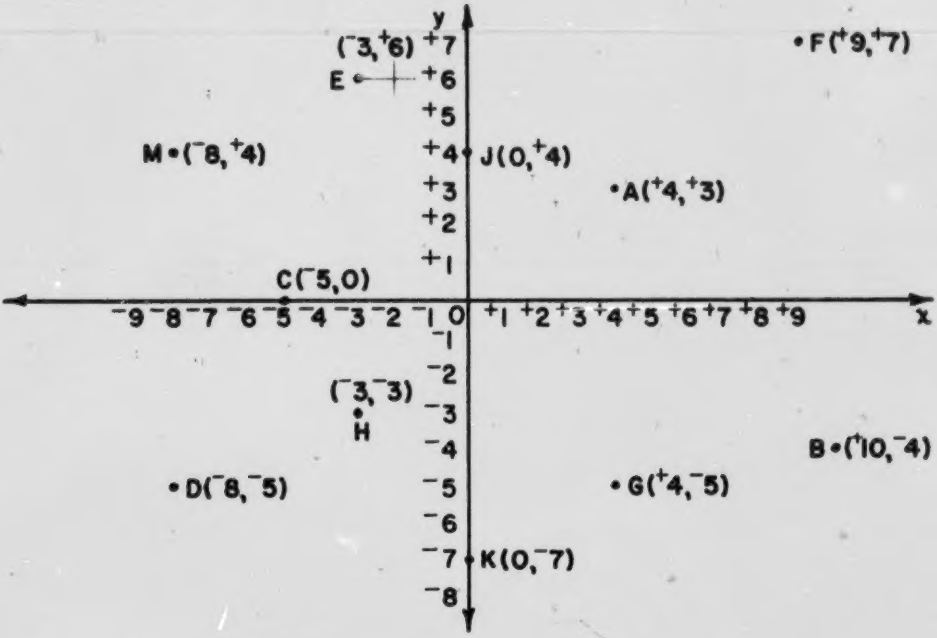
Label the 0-point and several points on each axis.

When you graph an ordered pair, label the point with the ordered pair.

Label the x-axis and the y-axis on your graph paper for Exercise 2.

Exercise Set 3

1. Write as ordered pairs the coordinates of the labeled points.


y
x
(⁻3,⁺6)
E
F(⁺9,⁺7)
M•(⁻8,⁺4)
J(0,⁺4)
•A(⁺4,⁺3)
C(⁻5,0)
(⁻3,⁻3)
H
B•(⁺10,⁻4)
•D(⁻8,⁻5)
•G(⁺4,⁻5)
K(0,⁻7)
⁺7 ⁺6 ⁺5 ⁺4 ⁺3 ⁺2 ⁺1
⁻1 ⁻2 ⁻3 ⁻4 ⁻5 ⁻6 ⁻7 ⁻8
⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7 ⁺8 ⁺9

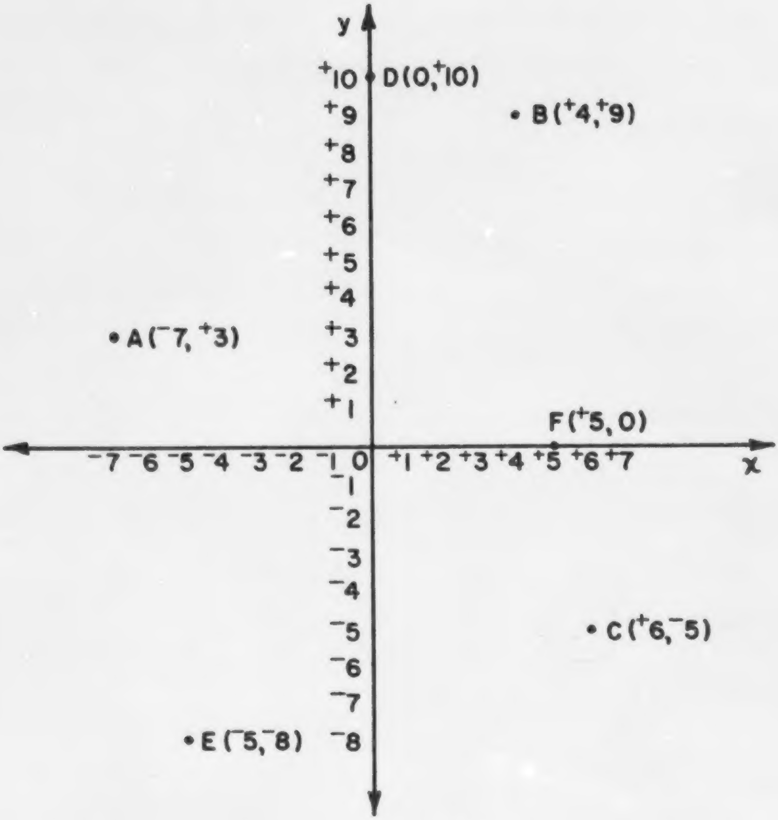
2. Graph the following ordered pairs. Use graph paper and label the x-axis and the y-axis. Label each point.

A($^{-}7$, $^{+}3$)	D(0, $^{+}10$)	(For solution see TC page 443.)
B($^{+}4$, $^{+}9$)	E($^{-}5$, $^{-}8$)	
C($^{+}6$, $^{-}5$)	F($^{+}5$, 0)	

3. Can you write an ordered pair of numbers to tell the location of your town on the earth's surface? What would you use for number lines? (The equator and the prime meridian are the coordinate lines.)

Answers: Exercise Set 3

Exercise 2


y
x
D(0, ⁺10)
B(⁺4, ⁺9)
A(⁻7, ⁺3)
F(⁺5, 0)
C(⁺6, ⁻5)
E(⁻5, ⁻8)
⁺10 ⁺9 ⁺8 ⁺7 ⁺6 ⁺5 ⁺4 ⁺3 ⁺2 ⁺1
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7
⁻1 ⁻2 ⁻3 ⁻4 ⁻5 ⁻6 ⁻7 ⁻8

Exploration

(X-Coordinates and Y-Coordinates)

You have been using two perpendicular number lines with the same zero point. We called these number lines the x-axis and the y-axis. These lines help you identify the point in a plane which is the graph of an ordered pair of numbers.

We say "ordered pair" because the order in which the two numbers are named is important. The point named by the pair ($^{-}3$, $^{+}6$) is a different point from the one named by the pair ($^{+}6$, $^{-}3$).

The first number in an ordered pair, which tells how far the point is to the right or left of the y-axis, is called the x-coordinate. The number which tells the distance of the point above or below the x-axis is called the y-coordinate.

1. a) Name the x-coordinate of each point graphed in Exercise 1 of Exercise Set 3.

A ($^{+}4$)
B ($^{+}10$)
C ($^{-}5$)
D ($^{-}9$)
E ($^{-}3$)
F ($^{+}9$)
G ($^{+}4$)
H ($^{-}3$)
J (0)
K (0)
M ($^{-}8$)

b) Name the y-coordinate of each of these points.

A ($^{+}9$)
B ($^{-}8$)
C (0)
D ($^{-}5$)
E ($^{+}6$)
F ($^{+}7$)
G ($^{-}5$)
H ($^{-}3$)
J ($^{+}4$)
K ($^{-}7$)
M ($^{+}4$)

2. Write as an ordered pair the coordinates of these points:

A: x-coordinate is $^{+}3$, y-coordinate is $^{-}7$. A($^{+}3$, $^{-}7$)

B: y-coordinate is $^{+}10$, x-coordinate is $^{-}3$. B($^{-}3$, $^{+}10$)

C: x-coordinate is 0, y-coordinate is $^{+}4$. C(0, $^{+}4$)

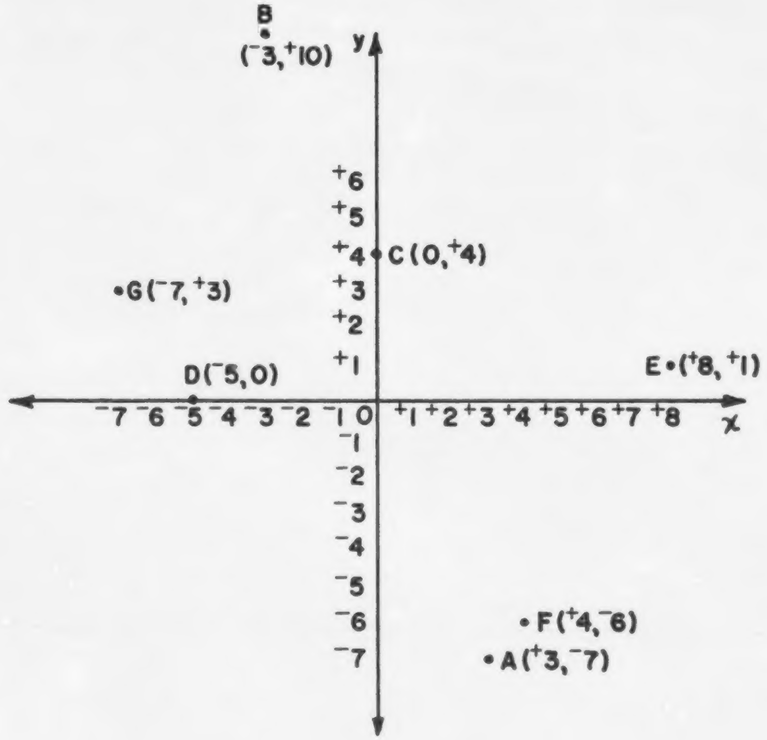
D: x-coordinate is $^{-}5$, y-coordinate is 0. D($^{-}5$,0)

E: x-coordinate is $^{+}8$, y-coordinate is $^{+}1$. E($^{+}8$, $^{+}1$)

F: y-coordinate is $^{-}6$, x-coordinate is $^{+}4$. F($^{+}4$, $^{-}6$)

G: x-coordinate is $^{-}7$, y-coordinate is $^{+}3$. G($^{-}7$, $^{+}3$)

3. Graph the points whose x- and y-coordinates are given in Exercise 2. Label each with its letter name and its ordered pair.



P265

Exercise Set 4

1. a) Graph the points whose coordinates are given below. Label each point with its letter name and its ordered pair. (For graph see TC page 447)

A: $(^-1, ^+7)$

B: $(^-4, 0)$

C: $(^+2, 0)$

b) Draw $\overline{AB}$ and $\overline{AC}$. The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is a triangle. What kind of triangle is it? (isosceles)

c) The base of (isosceles) triangle ABC is on the (x) axis.

2. a) Graph these points. Label each point with its letter name and its ordered pair.

D: x-coordinate $^-3$, y-coordinate $^+3$

E: x-coordinate $^-1$, y-coordinate $^-3$

F: x-coordinate $^+5$, y-coordinate $^-1$

G: x-coordinate $^+3$, y-coordinate $^+5$

(For graph see T.C. page 447)

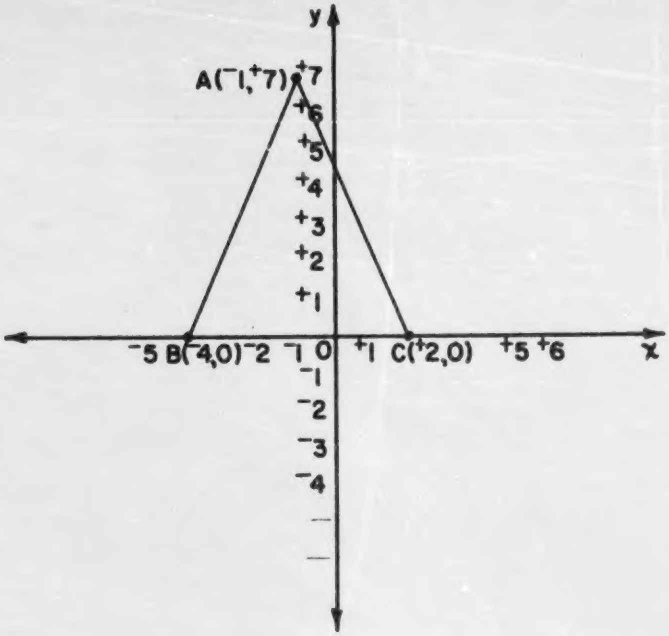
b) Draw $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GD}$. Which of these names describe the figure DEFG? (1, 2, 3, 5, 7)

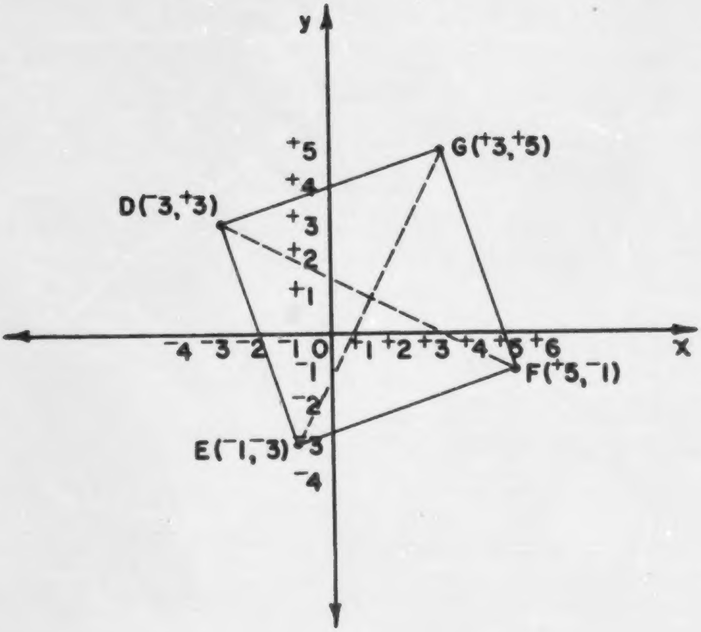
1) Simple closed curve
2) Polygon
3) Quadrilateral
4) Square region
5) Square
6) Isosceles triangle
7) Rectangle
8) Union of four angles

c) Draw $\overline{DF}$ and $\overline{EG}$. What are the coordinates of their intersection? $(^+1, ^+1)$

Answers: Exercise Set 4

Exercises 1-2


y
A(⁻1,⁺7)
⁺7
⁺6
⁺5
⁺4
⁺3
⁺2
⁺1
⁻5 B(⁻4,0) ⁻2 ⁻1 0 ⁺1 C(⁺2,0) ⁺5 ⁺6
x
⁻1
⁻2
⁻3
⁻4


y
G(⁺3,⁺5)
⁺5
⁺4
D(⁻3,⁺3)
⁺3
⁺2
⁺1
⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6
x
F(⁺5,⁻1)
⁻1
⁻2
E(⁻1,⁻3)
⁻3
⁻4

3. a) Write three different ordered pairs in which the second number is 0. (Answers will vary. Some examples are $(^{+}3,0)$, $(^{-}4,0)$, etc.)

   b) Graph these ordered pairs. (Graphs will vary.)

   c) Where do the three points lie? (They will lie on the x-axis)

   d) Write three different ordered pairs in which the first number is 0. (Answers will vary. Some examples are $(0,^{+}5)$, $(0,^{-}2)$ etc.)

   e) Graph these ordered pairs using the same axes you used for b.

   f) Where do these points lie? (They will lie on the y-axis.)

4. a) Any point whose x-coordinate is 0 lies on the (y-axis) ?.

   b) Any point whose y-coordinate is 0 lies on the (x-axis) ?.

5. a) What are the coordinates of the intersection of the x-axis and the y-axis? (0,0)

   b) What special name is given to the point of intersection of the x-axis and y-axis? (origin)

IV. USING COORDINATES TO FIND MEASURES OF SEGMENTS

Objective: To lead pupils to find lengths of segments, first by counting unit segments when either the x-coordinate or the y-coordinate remains constant, and second, to find the length by computation.

Vocabulary: unit segments

Materials: graph paper, pencil, squared chalkboard, straightedge

Teaching Procedure:

Before the children open their books, graph a set of ordered pairs on the squared chalkboard. These ordered pairs should be chosen so that either the x-coordinate or the y-coordinate is the same for all pairs. Help the children to recognize that the points lie along a line segment. Notice the position of the line segment. Is it to the right or left of the y-axis or above or below the x-axis? The teacher might desire to repeat this development with the segment parallel to the other axis. With practice children can visualize the location and position of a line segment by looking at a set of ordered pairs.

Now give the children the opportunity to follow the exploration. They should learn to find the measure of segments (in units), if the x-coordinate or y-coordinate is the same, by counting and then by finding the difference by computation. This activity will direct them to the generalization stated in Exercise 7.

The area of the quadrilateral region ABCD in Exercise 5 of Exercise Set 5 is the sum of the areas of all the right triangular regions plus the area of the rectangular region EFGH. Recall that the area of the rectangular region is obtained by multiplying the measures of the length and the width. The area of a right triangular region is found from the formula $\frac{b \times h}{2}$ where b and h are the measures of the segments which determine the right angle. In each right triangle the base may be considered as either side of the right angle. For example, in CGB if $\angle G$ is the right angle, side GC or side GB may be considered as the base. The area is in square units. Some of the children may want to do other exercises similar to this to see that it is possible to find the area of almost any polygonal region by dissecting it into rectangular and right triangular regions.

USING COORDINATES TO FIND MEASURES OF SEGMENTS

Exploration

1. a) Write 5 ordered pairs which have the same x-coordinate (do not use 0) and different y-coordinates. [Answers will vary. Some examples are $A(^+3, ^+6)$, $B(^+3, ^+3)$; $C(^+3, ^-1)$, $D(^+3, ^-4)$, etc.]

   b) Graph the points for the ordered pairs. (See T.C. page 452)

   c) Are all five points on the same line? (yes)

2. a) Write 5 ordered pairs which have different x-coordinates and the same y-coordinates (do not use 0). [Answers will vary. Some examples are $A(^-3, ^+5)$, $B(^+4, ^+5)$, $C(^+1, ^+5)$, $D(^+8, ^+5)$, etc.]

   b) Graph the points for the ordered pairs. (See T.C. page 452.)

   c) Are all five points on the same line? (yes)

3. What do you notice about the lines suggested in 1 c) and 2 c)? (If the x-coordinates are the same, the line will be vertical. If the y-coordinates are the same, they lie on a horizontal line.)

4. a) Graph the points $R(^-2, ^+7)$ and $S(^-2, ^+3)$ (For graph see T.C. page 452)

   b) What is the measure of $\overline{RS}$ in unit segments? (4 units)

   c) Could you find the measure of $\overline{RS}$, without counting unit segments, by using the y-coordinates? (Yes, $^+7 - ^+3 = ^+4$ The length is 4 units)

5. a) Graph the points $A(^-3, ^-5)$ and $B(^+4, ^-5)$ (For graph, see T.C. page 452.)

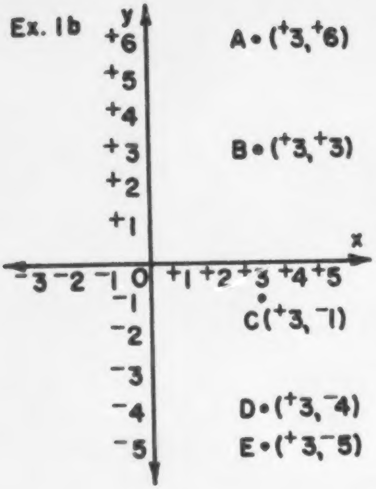
   b) Subtract $^-3$ from $^+4$. $^+4 - ^-3 = ?$ ($^+7$)

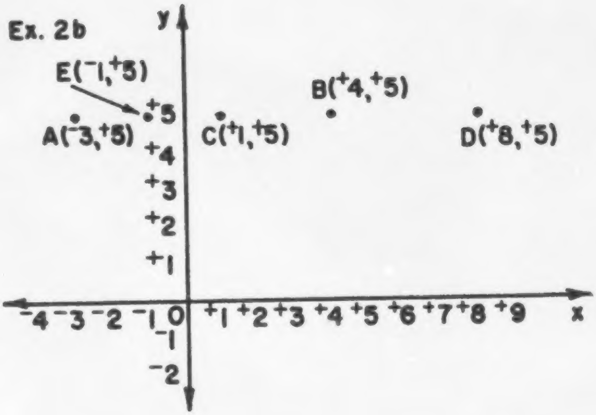
   c) Subtract $^+4$ from $^-3$. $^-3 - ^+4 = ?$ ($^-7$)

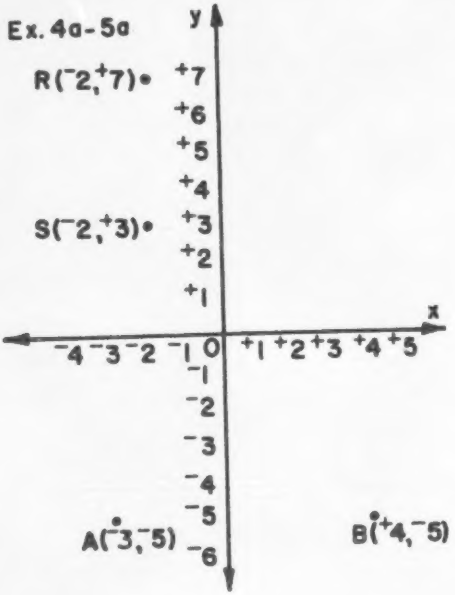
   d) Does your answer to either b) or c) tell you the length of $\overline{AB}$? (Yes, they both tell the length of $\overline{AB}$. It is 7 units long.)

Answers: Exploration

Exercises 1b-2a-4a-5a







6. a) Consider the points C($^{+}105$, $^{+}58$) and D($^{+}105$, $^{+}69$). Without graphing C and D, can you find the length of $\overline{CD}$? (yes, $\overline{CD}$ is 11 units in length. $^{+}58 - {}^{+}69 = {}^{-}11$ or $^{+}69 - {}^{+}58 = {}^{+}11$ This is possible because x-coordinates are equal.)

   b) What is the length of $\overline{RS}$ if R has coordinates ($^{-}3$, $^{-}579$) and S has coordinates ($^{-}3$, $^{-}468$)? ($\overline{RS}$ is 111 units in length. $^{-}579 - {}^{-}468 = {}^{-}111$; $^{-}468 - {}^{-}579 = {}^{+}111$ This is possible because the x-coordinates are the same)

7. From your observations in Exercises 1-6, complete this sentence:

   To find an integer which tells the measure, in units, of the segment between two points:

   a) if the x-coordinates are the same (subtract) one (y) coordinate from the other.

   b) if the y-coordinates are the same subtract one (x) coordinate from the other.

Exercise Set 5

1. Here are names of points and the coordinates of each:

$A(^+5, ^-7)$

$B(^-2, ^+3)$

$C(^-2, ^-7)$

$D(^+2, ^-7)$

$E(^+5, ^+3)$

$F(^-8, ^-1)$

a) List the pairs of points with the same x-coordinate.
[$B(^-2, ^+3)$ and $C(^-2, ^-7)$; $A(^+5, ^-7)$ and $E(^+5, ^+3)$]
Then find the length of the segment joining each pair.
($\overline{BC}$ = 10 units ; $\overline{AE}$ = 10 units)

b) Find the pairs of points with the same y-coordinate.
[$A(^+5, ^-7)$ $C(^-2, ^-7)$; $A(^+5, ^-7)$ $D(^+2, ^-7)$; $C(^-2, ^-7)$ $D(^+2, ^-7)$; $B(^-2, ^+3)$, $E(^+5, ^+3)$]
Then find the length of the segment joining each pair.
($\overline{AC}$ = 7 units , $\overline{AD}$ = 3 units , $\overline{CD}$ = 4 units , $\overline{BE}$ = 7 units)

c) Check your answers by graphing the ordered pairs and counting unit segments. (See T.C. page 455.)

2. a) Graph this set of ordered pairs of integers and label the points of the graph. $\{A(^+2, ^+9), B(^+2, ^-2), C(^+7, ^-2), D(^+7, ^+9)\}$ (See TC page 455)

b) Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.
What kind of quadrilateral is ABCD? (rectangle)

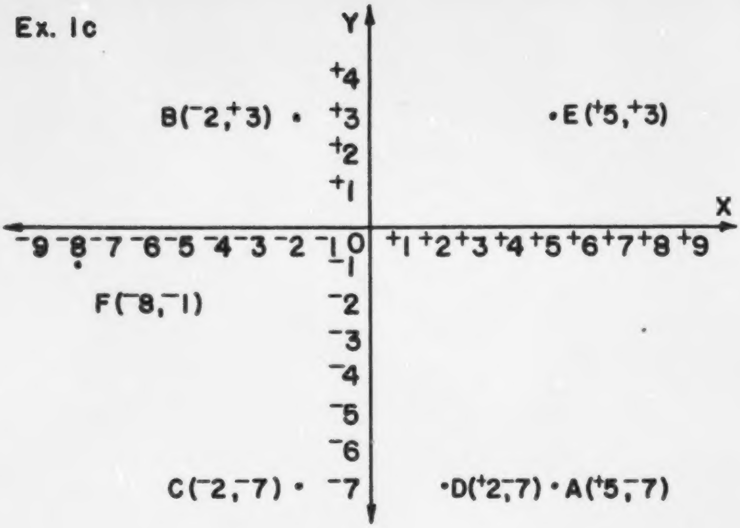
c) Find the lengths of $\overline{AB}$ and $\overline{BC}$. (Don't count, subtract coordinates.) ($\overline{AB}$ = 11 units ; $\overline{BC}$ = 5 units)

d) What is the area of region ABCD? (55 square units)

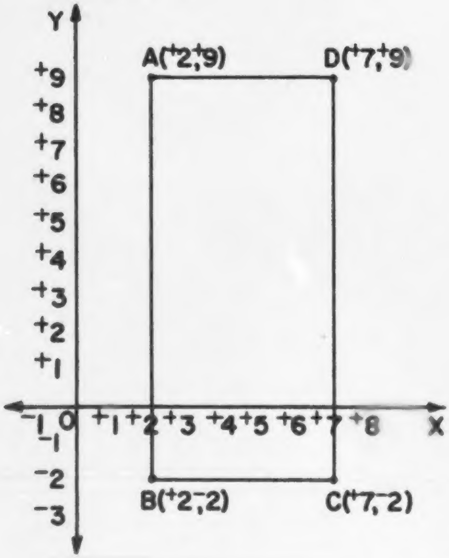
Answers: Exercise Set 5

Exercises 1c-2a-3a-3b

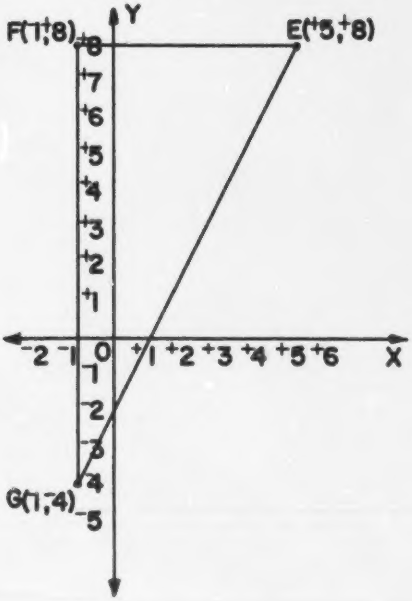
Ex. 1c


Y
X
B(⁻2,⁺3)
E(⁺5,⁺3)
F(⁻8,⁻1)
C(⁻2,⁻7)
D(⁺2,⁻7)
A(⁺5,⁻7)
⁺4 ⁺3 ⁺2 ⁺1
⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7 ⁺8 ⁺9
⁻1 ⁻2 ⁻3 ⁻4 ⁻5 ⁻6 ⁻7

Exercise 2


Y
X
A(⁺2,⁺9)
D(⁺7,⁺9)
B(⁺2,⁻2)
C(⁺7,⁻2)
⁺9 ⁺8 ⁺7 ⁺6 ⁺5 ⁺4 ⁺3 ⁺2 ⁺1
⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7 ⁺8
⁻1 ⁻2 ⁻3

Exercise 3

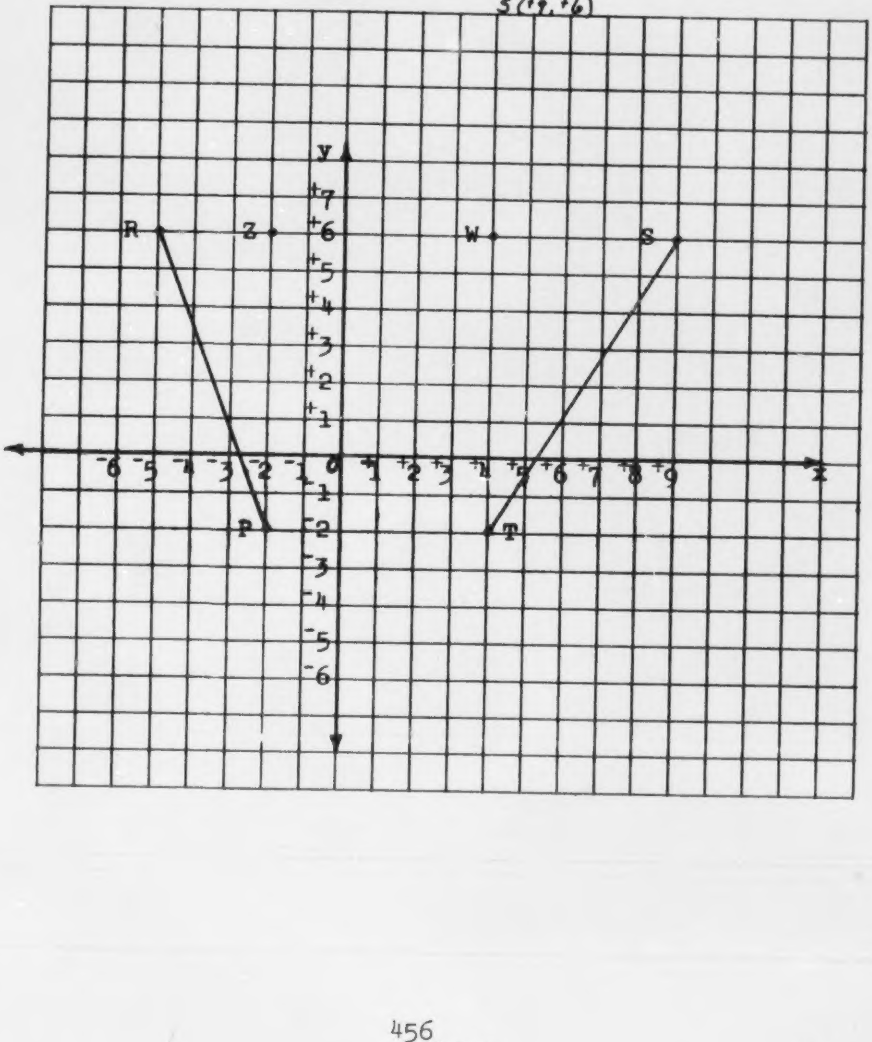

Y
X
F(1,⁺8)
E(⁺5,⁺8)
G(1,⁻4)
⁺8 ⁺7 ⁺6 ⁺5 ⁺4 ⁺3 ⁺2 ⁺1
⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6
⁻1 ⁻2 ⁻3 ⁻4 ⁻5

3. a) Graph these points E($^+5$, $^+8$), F($^-1$, $^+8$), G($^-1$, $^-4$).
(For graph, see T.C. page 465)

b) Draw $\overline{EF}$, $\overline{FG}$, $\overline{EG}$. What segments are the base and height of Δ EFG? ($\overline{EF}$ could be the base and $\overline{FG}$ could be the height.)

c) Find the area of Δ EFG. (Area of Δ EFG = $\frac{1}{2}$ × base × height
base = 6 units
height = 12 units
Area of Δ EFG = $\frac{1}{2}$ × 6 × 12 = 36 square units)

4. a) Write on your paper the coordinates of each labeled point in the figure below.
R($^-5$, $^+6$)
Z($^-2$, $^+6$)
W($^+4$, $^+6$)
S($^+9$, $^+6$)
P($^-2$, $^-2$)
T($^+4$, $^-2$)



b) Find: 1) a set of four labeled points with the same y-coordinate. (R, Z, W, S)

2) a set of two labeled points with the same y-coordinate. (P, T)

3) two sets of two labeled points with the same x-coordinate. (Z and P; W and T)

c) Find the lengths of these line segments: $\overline{RZ}$, $\overline{ZW}$, $\overline{WS}$, $\overline{ZP}$, $\overline{WT}$, $\overline{PT}$.

($\overline{RZ} = 3$ units
$\overline{ZW} = 6$ units
$\overline{WS} = 5$ units
$\overline{ZP} = 8$ units
$\overline{WT} = 8$ units
$\overline{PT} = 6$ units)

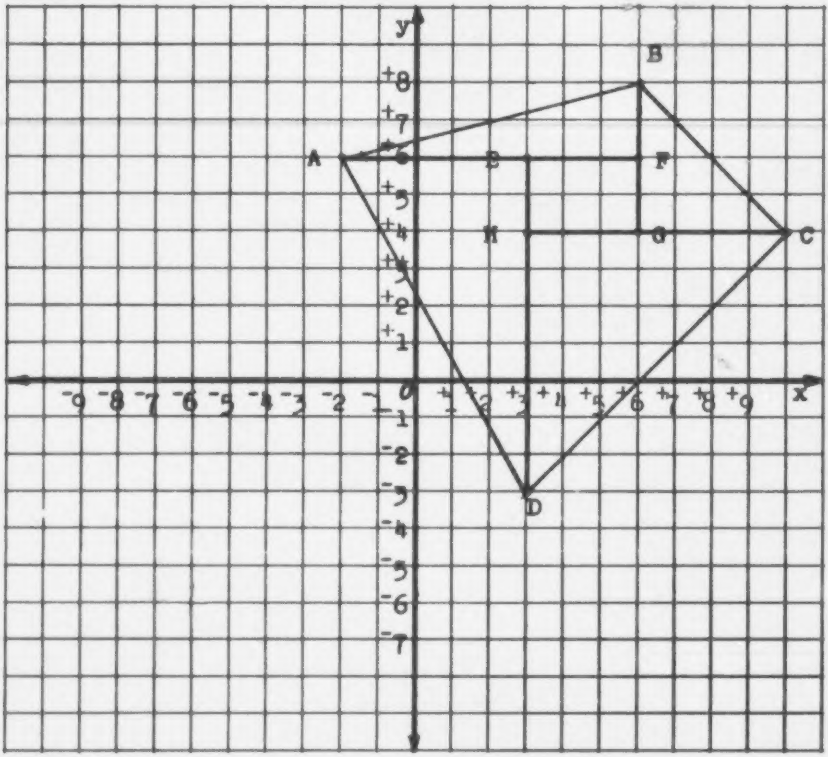
d) Name two triangles and a rectangle in the figure, and find the area of each region.

(area of $\triangle RZP = 12$ sq units.
area of $\triangle WST = 20$ sq units
area of rectangle ZWTP = 48 sq units)

e) What is the area of the region bounded by quadrilateral RSTP?

(Area of quadrilateral RSTP = 12 + 20 + 48
= 80 sq units)

5. a) Write the coordinates of each labeled point in the figure.



$A(^{-}2, ^{+}6)$
$B(^{+}6, ^{+}8)$
$C(^{+}10, ^{+}4)$
$D(^{+}3, ^{-}3)$
$E(^{+}3, ^{+}6)$
$F(^{+}6, ^{+}6)$
$G(^{+}6, ^{+}4)$
$H(^{+}3, ^{+}4)$

b) Figure ABCD is a ___?___ . (quadrilateral or polygon)

c) What set of three points have the same x-coordinate? Can you find another set? (E, H, and D ; B, F, and G)

d) What three labeled points have the same y-coordinate? Can you find another set? (A, E, and F ; H, G, and C)

e) Find the lengths of base and altitude of each right triangle with labeled vertices. (There are four.)

△ AED	△ AFB	△ BCG	△ HCD
$\overline{AE}$ = 5 units	$\overline{AF}$ = 8 units	$\overline{GC}$ = 4 units	$\overline{HC}$ = 7 units
$\overline{ED}$ = 9 units	$\overline{BF}$ = 2 units	BG = 4 units	$\overline{HD}$ = 7 units

f) Find the area of each triangular region and the area of the rectangular region. (Area of $\triangle$ AED = $22\frac{1}{2}$ sq. units
Area of $\triangle$ AFB = 8 sq. units
Area of $\triangle$ BCG = 8 sq. units
Area of $\triangle$ HCD = $24\frac{1}{2}$ sq. units
Area of rectangle EFGH = 6 sq. unit)

g) Find the area of the region ABCD.
(Area of the region ABCD = $22\frac{1}{2} + 8 + 8 + 24\frac{1}{2} + 6$
= 69 square unit)

6. Battleship Game 1: Here is a game you might find interesting. It requires two people (or two "sides") to play it. Here are the rules. They are stated for two players. If there are two "sides" with several players on each side then the "sides" play alternately with each player on a side playing in turn. The game can also be played on a piece of paper, rather than on a chalkboard.

a) Draw (on the chalkboard) x- and y-axes and mark each axis with numerals from $^-10$ to $^+10$.

b) One player marks (with white chalk) 10 points (battleships) each having coordinates which are integers. Do not label the points with their coordinates.

c) The opponent player marks (with colored chalk) 10 new points (battleships) for his side, each point having coordinates which are integers. Do not label the points with their coordinates.

d) The first player calls out an ordered pair of integers. If there is a point (battleship) marked with these coordinates then the marking is erased (battleship sunk). (Sometimes a player makes a mistake and sinks one of his own battleships.) If there is no point marked with these coordinates then the opponent has his turn.

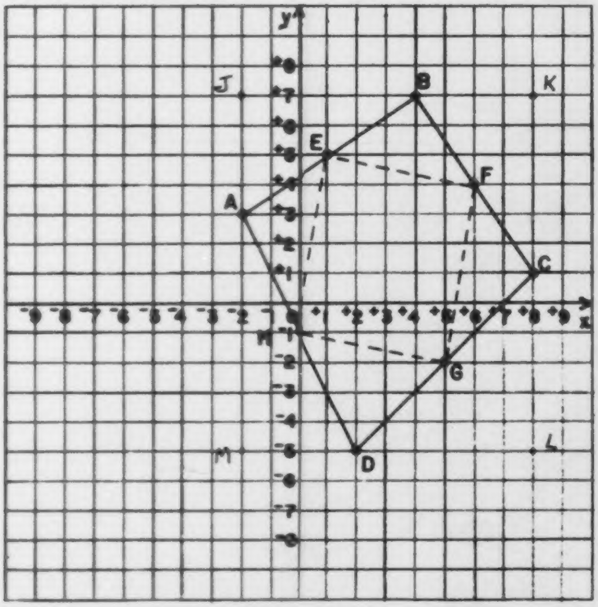
e) The second player now has his turn to carry out Rule d.

f) Players now play alternately.

g) The player whose battleships are all sunk first (after each player has had the same number of turns) loses the game. If all battleships of both sides are sunk, then the game is a tie.

BRAINTWISTER

1. a) Write the coordinates of each labeled point in the figure:


y
x
A
B
C
D
E
F
G
H
J
K
L
M
⁺8
⁺7
⁺6
⁺5
⁺4
⁺3
⁺2
⁺1
0
⁻1
⁻2
⁻3
⁻4
⁻5
⁻6
⁻7
⁻8
⁻9
⁻8
⁻7
⁻6
⁻5
⁻4
⁻3
⁻2
⁻1
⁺1
⁺2
⁺3
⁺4
⁺5
⁺6
⁺7
⁺8
⁺9

$A(^{-}2,^{+}3)$
$B(^{+}4,^{+}7)$
$C(^{+}8,^{+}1)$
$D(^{+}2,^{-}5)$
$E(^{+}1,^{+}5)$
$F(^{+}6,^{+}4)$
$G(^{+}5,^{-}2)$
$H(0,^{-}1)$

b) Figure ABCD is a (quadrilateral or polygon).

c) Draw $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{HE}$.

d) Figure EFGH is a (parallelogram).

e) On the graph mark:

(1) A point J whose x-coordinate is the same as the x-coordinate of point A and whose y-coordinate is the same as the y-coordinate of point B.

(2) A point K whose x-coordinate is the same as the x-coordinate of the point C and whose y-coordinate is the same as the y-coordinate of point B.

(3) A point L whose x-coordinate is the same as the x-coordinate of point C and whose y-coordinate is the same as the y-coordinate of point D.

(4) A point M whose x-coordinate is the same as the x-coordinate of point A and whose y-coordinate is the same as the y-coordinate of point D.

f) Figure JKLM is a (rectangle).

g) Compute the area of the region JKLM. (120 sq units)

h) Compute the lengths of the base and altitude of each of the right triangles, ΔAJB, ΔBKC, ΔCLD, and ΔDMA.

[$\overline{JB}$ = 6 unit, $\overline{JA}$ = 4 unit; $\overline{BK}$ = 4 unit, $\overline{KC}$ = 6 unit; $\overline{DL}$ = 6 unit, $\overline{CL}$ = 6 unit; $\overline{MD}$ = 4 unit, $\overline{MA}$ = 8 unit]

i) Compute the areas of each of the triangular regions whose sides are the triangles of Exercise h. (area of ΔDMA = 16 sq unit, area of ΔAJB = 12 unit, area of ΔBKC = 12 unit, area of CLD = 18 unit.)

j) What is the area of the polygon region ABCD?

[area of the polygon ABCD = 120 − (16 + 12 + 12 + 18)
= 120 − 58
= 62 square unit]

V. CHANGING COORDINATES

Objective: To show that a change in one of the coordinates of each point of a figure, while the other remains constant, will "move" the figure to the left or right or up or down, and that the direction and distance may be predicted by observing the amount of the change and whether it is in the x- or the y-coordinate.

Materials: graph paper, pencils, straightedge, chalkboard. (Opaque projector or overhead projector if available)

Vocabulary: changing coordinates

Teaching Procedure:

> To introduce this section the teacher might illustrate the relation between two figures if the coordinates of the points of the second figure are obtained from those of the first by some simple change. This can be done with a line or a figure on the graph by the use of a chalkboard or an opaque or overhead projector. Following the Exploration as given would be very effective using one of these visual aids. The climax to this demonstration would be exercises 9 and 10. In using an opaque projector the steps could be shown by using overlays, each successive step being shown on a separate sheet of graph paper. The overlay sheets should be prepared in advance of the lesson. In using an overhead projector or the chalkboard the points are identified as the demonstration proceeds.

CHANGING COORDINATES

Exploration

1. Suppose that P is the point $(^+1, ^+2)$ and R is the point $(^+3, ^+6)$.

   How many segments are there with P and R as endpoints? (one)

2. Graph the ordered pairs P and R in Exercise 1 and draw $\overline{PR}$.

3. Is $(^+2, ^+4)$ on $\overline{PR}$? (yes) If so, mark and label it S.

4. Graph the ordered pair $T(^+5, ^+2)$ and draw $\overline{RT}$.

5. Is the point $(^+4, ^+4)$ on $\overline{RT}$? (yes) Mark and label it W.

6. Draw $\overline{SW}$.

7. What letter does the figure look like? (Capital A)

8. The set of ordered pairs you have graphed is:

   $\{P(^+1, ^+2), S(^+2, ^+4), R(^+3, ^+6), W(^+4, ^+4)$, and $T(^+5, ^+2)\}$.

   Form a new set of ordered pairs by changing the coordinates of these pairs as follows: Add $^+7$ to each x-coordinate and,
   add 0 to each y-coordinate.

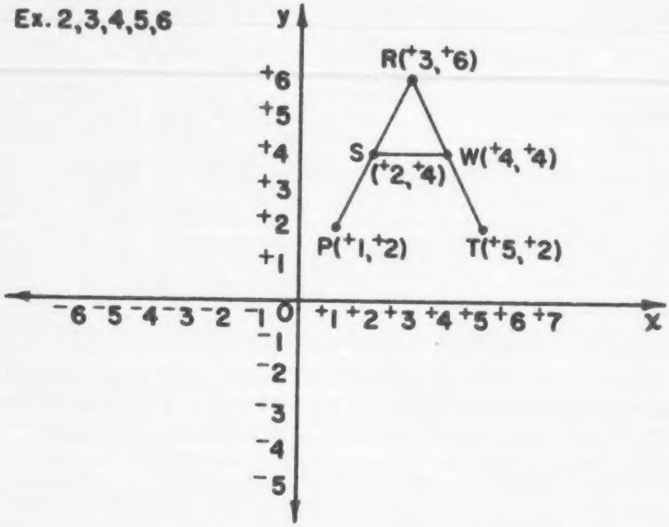
   Name the corresponding new points, A, B, C, D, and E.

9. a) Graph and label the set of ordered pairs you found in Exercise 8 and draw the segments AC, CE, and BD.

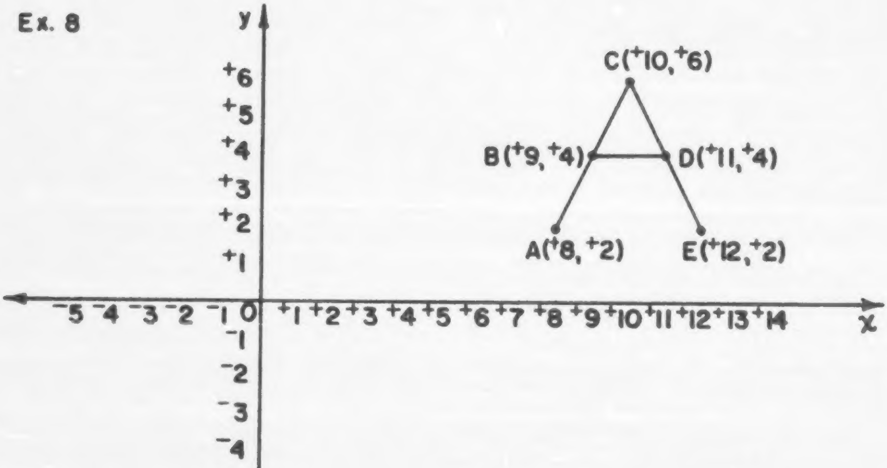
   b) What does this new figure look like? (Capital A)

   c) How is the new figure related to the old figure?
   (The figure was moved 7 units to the right. The figure is congruent to the first.)

Answers: Exploration
Exercises 2-3-4-5-6-8



Ex. 2,3,4,5,6
y
R(+3, +6)
S
(+2, +4)
W(+4, +4)
P(+1, +2)
T(+5, +2)
+6
+5
+4
+3
+2
+1
−6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7
x
−1
−2
−3
−4
−5



Ex. 8
y
C(+10, +6)
B(+9, +4)
D(+11, +4)
A(+8, +2)
E(+12, +2)
+6
+5
+4
+3
+2
+1
−5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12 +13 +14
x
−1
−2
−3
−4
−5
−6

Exercise Set 6

1. a) Graph and label this set of points: (For graph, see TC 466)

   $\{A(^{+}5, ^{-}1), B(^{+}5, ^{+}5), C(^{-}4, ^{+}5), D(^{-}4, ^{-}1)\}$

   b) Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.

   c) What kind of polygon is ABCD? (rectangle, quadrilateral)

2. a) Form a new set of ordered pairs by subtracting 0 from each x-coordinate and subtracting $^{+}8$ from each y-coordinate of the set of pairs of 1a.

   b) Plot this set of points. (Call the corresponding points E, F, G, H.) (For graph, see TC 466)

   c) Draw $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HE}$.

   d) Is EFGH congruent to ABCD? (Yes)

   e) How is the new figure we got by changing the y-coordinates related to the old figure? (Moved the figure down 8 units. It is congruent to the first figure.)

3. a) Graph this set of ordered pairs. Label the points.

   $\{A(^{-}3, ^{+}7), B(^{+}1, ^{+}9), C(^{+}1, ^{+}4), D(^{+}3, 0)\}$

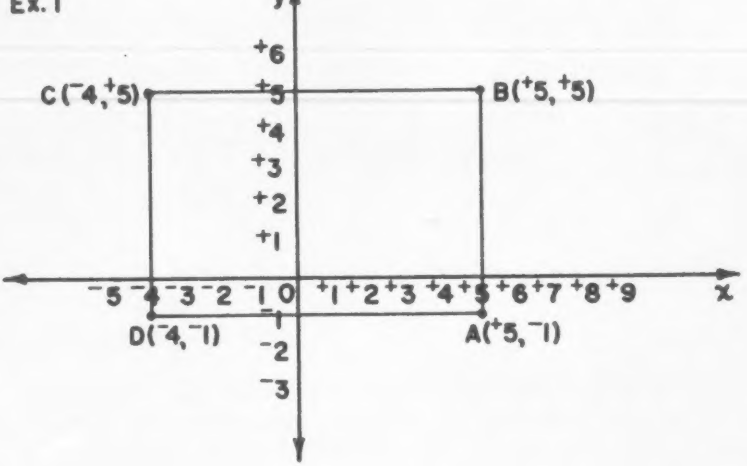
   (For graph, see page 468)

   b) Draw $\overline{AC}$, $\overline{BC}$, $\overline{CD}$. What letter does the union of these segments look like? (Capital Y)

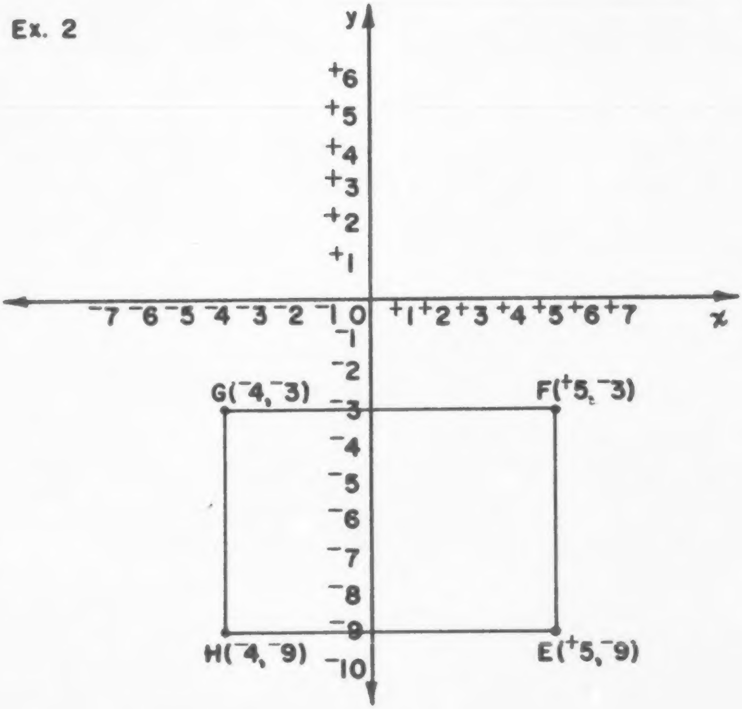
Answers: Exercise Set 6

Exercises 1-2

Ex. 1


y
+6
C(⁻4,⁺5)
+5
B(⁺5,⁺5)
+4
+3
+2
+1
⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7 ⁺8 ⁺9
x
⁻1
D(⁻4,⁻1)
A(⁺5,⁻1)
⁻2
⁻3

Ex. 2


y
+6
+5
+4
+3
+2
+1
⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 ⁺1 ⁺2 ⁺3 ⁺4 ⁺5 ⁺6 ⁺7
x
⁻1
⁻2
G(⁻4,⁻3)
F(⁺5,⁻3)
⁻3
⁻4
⁻5
⁻6
⁻7
⁻8
⁻9
H(⁻4,⁻9)
E(⁺5,⁻9)
⁻10

4. a) Form a new set of ordered pairs {E, F, G, H} from the old set of Exercise 3a as follows: Add $^+3$ to the x-coordinate and subtract $^+6$ from the y-coordinate.

   b) Graph this set of ordered pairs. Draw $\overline{EG}$, $\overline{FG}$ and $\overline{GH}$. (For graph, see page 468)

   c) How does the shape and position of this figure compare with the one for Exercise 3? (The second figure is congruent to the first. The figure moved 3 units to the right and 6 units down.)

5. Suppose you wanted to "move" the figure of Exercise 3 five units to the left. What change would you make in the coordinates? (Subtract a $^+5$ from all x-coordinates.)

6. What change in the coordinates would "move" the figure three units up? (Add a $^+3$ to all y-coordinates.)

7. Battleship Game 2: The rules of this game are the same as those for Battleship Game 1 (see end of Exercise Set 5) except that Rule d is replaced by:

   d*) (1) The first player calls out something like "2 units exactly to left of $(^+1, ^+3)$". If there is a point (battleship) marked with the coordinates $(^-1, ^+3)$ then the marking is erased (battleship sunk).

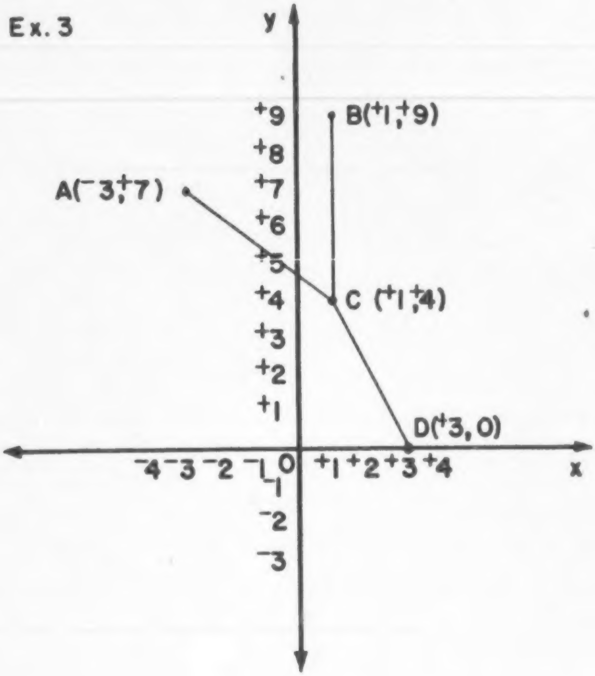
   (2) If the first pl[illegible]er said "2 units exactly to the right of $(^+1, ^+3)$" then the marking at $(^+3, ^+3)$ would be erased.

   (3) If the first player said "3 units exactly below $(^+1, ^+3)$," then the marking at $(^+1, 0)$ would be erased.

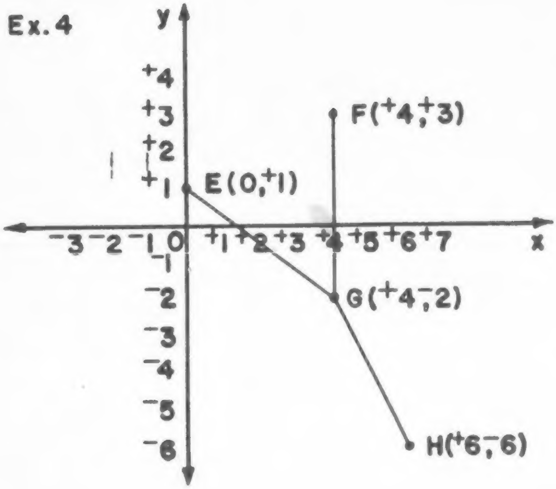
Answers: Exercise Set 6

Exercises 3-4

Ex. 3



Ex. 4



(4) if the first player said "3 units exactly above $(^+1, ^+3)$," then the marking at $(^+1, ^+6)$ would be erased.

(5) If there is no point marked at the coordinates described, then the opponent has his turn.

BRAINTWISTER: In the Exploration exercises you graphed ordered pairs and drew segments which formed the letter "A". Draw some other capital letter on a sheet of graph paper. Write on a separate sheet the coordinates of the endpoints of the segments that form the figure. Hand the instructions (but not the graph) to some other student. See if he can follow the instructions to obtain the same letter you have on your graph paper. (The solutions will vary.)

VI. GRAPHS OF SPECIAL SETS

Objective: To provide readiness for ideas of reflection and symmetry. To show that the graph of the points in which coordinates are in certain definite relationship lie on lines other than the axes.

Materials: pencils, graph paper, straightedge, squared chalkboard

Vocabulary: congruent

Teaching Procedure:

The Explorations in this section are in sufficient detail to follow. However, if more development is needed, it might be advantageous for the teacher to show the line segments whose endpoints are graphs of ordered pairs. First, a large chart could be completed at the chalkboard. This chart would be an aid in the discussion of the exploration.

At this stage of the unit it is hoped that most pupils will be able to graph ordered pairs with little or no assistance. Individual help may be necessary for a few pupils.

The Braintwisters are designed to develop the idea of similarity. For example, if a polygon has coordinates of all vertices doubled, tripled, or halved, then the resulting figure will be similar to the original one.

Braintwister Exercise 1 results in two similar triangles, the second has sides twice as long as the corresponding side in the first triangle. Corresponding angles are congruent although the corresponding sides are not.

Braintwister Exercise 2 may result in any polygon, depending upon the ordered pairs chosen by the pupil. In any case, the figure obtained by multiplying each coordinate by 3 will result in a figure that is similar to the first but which has corresponding line segments which are 3 times as long as those of the first figure.

GRAPHS OF SPECIAL SETS

Exploration

Suppose we have just the set $\{^+1, ^+2, ^+3, ^+4, ^+5\}$. You wish to write the set of all the ordered pairs of numbers such that both numbers are in this set.

1. Follow a system so you do not omit any pairs.

   First list in the first row all pairs which have $^+1$ as first number. Then list in a second row all which have $^+2$ as first number; and so on.

   Arrange your pairs in a chart as shown below. (For chart, see TC-472)

$(^+1, ^+1)$	$(^+1, ^+2)$	$(^+1, ?)$	$(^+1, ?)$	$(^+1, ?)$
$(^+2, ^+1)$	$(^+2, ?)$	$(?, ?)$	$(\quad)$	$(\quad)$
$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$
$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$
$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$	$(\quad)$

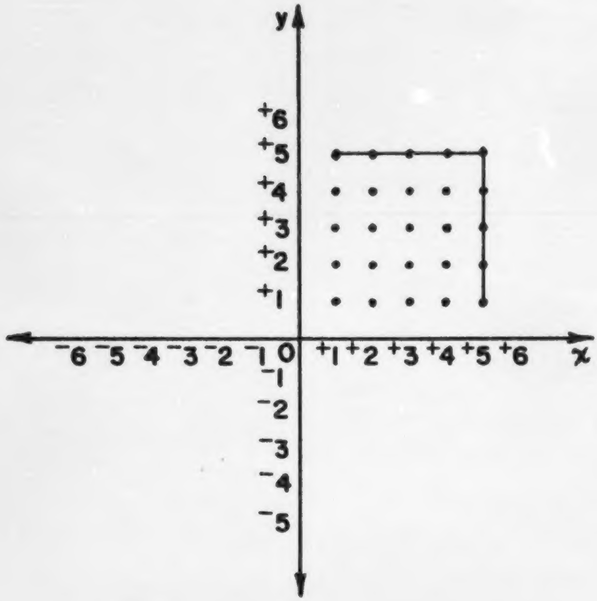
2) a) Does the chart contain all ordered pairs with both numbers in the set $\{^+1, ^+2, ^+3, ^+4, ^+5\}$? (yes)

   b) Where in the chart are all the pairs with $^+1$ as second number? (the first column) With $^+2$ as second number? (the second column) With $^+4$ as the first number? (the fourth row.)

Answers: Exploration and Exercise Set 7

Exercises 3b-4b-5b-6a-6b

Exercise 1



Exercise 1

$(^+1, ^+1)$	$(^+1, ^+2)$	$(^+1, ^+3)$	$(^+1, ^+4)$	$(^+1, ^+5)$
$(^+2, ^+1)$	$(^+2, ^+2)$	$(^+2, ^+3)$	$(^+2, ^+4)$	$(^+2, ^+5)$
$(^+3, ^+1)$	$(^+3, ^+2)$	$(^+3, ^+3)$	$(^+3, ^+4)$	$(^+3, ^+5)$
$(^+4, ^+1)$	$(^+4, ^+2)$	$(^+4, ^+3)$	$(^+4, ^+4)$	$(^+4, ^+5)$
$(^+5, ^+1)$	$(^+5, ^+2)$	$(^+5, ^+3)$	$(^+5, ^+4)$	$(^+5, ^+5)$

3. a) Find in the chart the ordered pairs in which the x-coordinate and y-coordinate are the same. List the pairs.
$[(^+1, ^+1), (^+2, ^+2), (^+3, ^+3), (^+4, ^+4), (^+5, ^+5)]$

   b) Graph these ordered pairs. (See TC-472)

4. a) Find in the chart the ordered pairs in which the second number is greater than the first number. List them.
$[(^+1, ^+2), (^+1, ^+3), (^+1, ^+4), (^+1, ^+5), (^+2, ^+3), (^+2, ^+4), (^+2, ^+5), (^+3, ^+4), (^+3, ^+5), (^+4, ^+5)]$

   b) Graph these ordered pairs, using a red crayon. Use the same axes you used for Exercise 3b. (See TC-472)

5. a) Find in the chart and list the ordered pairs in which the second number is smaller than the first.
$[(^+5, ^+4), (^+5, ^+3), (^+5, ^+2), (^+5, ^+1), (^+4, ^+3), (^+4, ^+2), (^+4, ^+1), (^+3, ^+2), (^+3, ^+1), (^+2, ^+1)]$

   b) Graph these ordered pairs on the same axes you used for Exercise 3b, using a green crayon. (See TC-472)

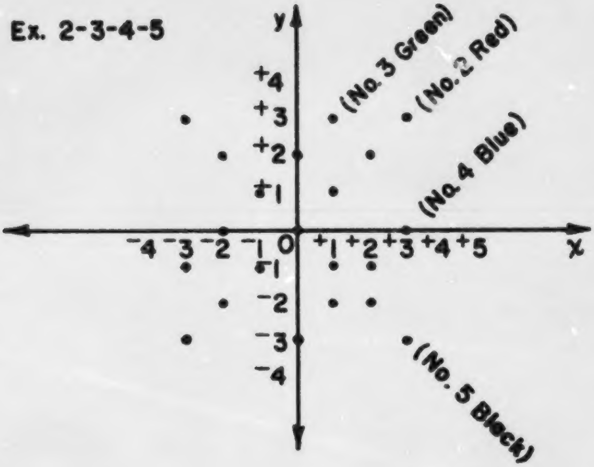
6. a) Find all the points whose y-coordinate is $^+5$. Draw in black the line segment through these points. (See TC-472)

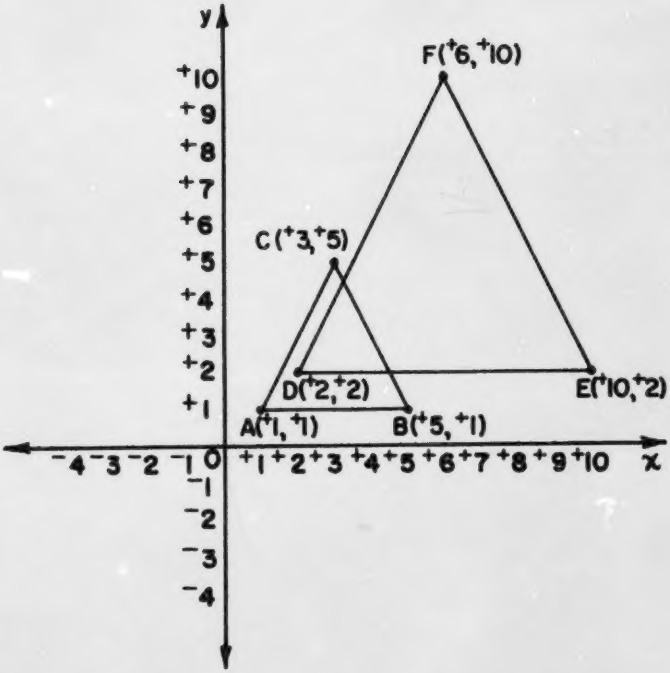
   b) Do the same for the points whose x-coordinate is $^+5$. (See TC-472)

$(^-3, ^-3)$	$(^-2, ^-3)$	$(^-1, ^-3)$	$(0, ^-3)$	$(^+1, ^-3)$	$(^+2, ^-3)$	$(^+3, ^-3)$
$(^-3, ^-2$	$(^-2, ^-2)$	$(^-1, ^-2)$	$(0, ^-2)$	$(^+1, ^-2)$	$(^+2, ^-2)$	$(^+3, ^-2)$
$(^-3, ^-1$	$(^-2, ^-1)$	$(^-1, ^-1)$	$(0, ^-1)$	$(^+1, ^-1)$	$(^+2, ^-1)$	$(^+3, ^-1)$
$(^-3, 0)$	$(^-2, 0)$	$(^-1, 0)$	$(0, 0)$	$(^+1, 0)$	$(^+2, 0)$	$(^+3, 0)$
$(^-3, ^+1)$	$(^-2, ^+1)$	$(^-1, ^+1)$	$(0, ^+1)$	$(^+1, ^+1)$	$(^+2, ^+1)$	$(^+3, ^+1)$
$(^-3, ^+2)$	$(^-2, ^+2)$	$(^-1, ^+2)$	$(0, ^+2)$	$(^+1, ^+2)$	$(^+2, ^+2)$	$(^+3, ^+2)$
$(^-3, ^+3)$	$(^-2, ^+3)$	$(^-1, ^+3)$	$(0, ^+3)$	$(^+1, ^+3)$	$(^+2, ^+3)$	$(^+3, ^+3)$

Answers: Exercise Set 7

Exercises 2-3-4-5-Braintwister 1


Ex. 2-3-4-5
y
x
+4
+3
+2
+1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 +1 +2 +3 +4 +5
(No. 3 Green)
(No. 2 Red)
(No. 4 Blue)
(No. 5 Black)


y
x
+10
+9
+8
+7
+6
+5
+4
+3
+2
+1
-1
-2
-3
-4
-4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10
F(+6, +10)
C(+3, +5)
D(+2, +2)
E(+10, +2)
A(+1, +1)
B(+5, +1)

Exercise Set 7

Use the numbers in this set: $\{^-3, ^-2, ^-1, 0, ^+1, ^+2, ^+3\}$.

1. Make a chart of all ordered pairs of numbers with both numbers in the set above. How many should there be? (49)

2. List the set of ordered pairs in the chart in which the two coordinates are the same. Graph this set in red. (See TC 474)
$[(^-3,^-3), (^-2,^-2), (^-1,^-1), (0,0), (^+1,^+1), (^+2,^+2), (^+3,^+3)]$

3. List the set of ordered pairs in which the second number is 2 greater than the first. (There should be five such pairs.)
$[(^-3,^-1), (^-2,0), (^-1,^+1), (0,^+2), (^+1,^+3)]$
Graph this set in green on the axes you used for Exercise 2. (See TC474)

4. List the set of ordered pairs in which the second number is 3 less than the first. (There should be four such pairs.)
$[(0,^-3), (^+1,^-2), (^+2,^-1), (^+3,0)]$
Graph this set in blue on the same axes. (See TC474)

5. List the set of ordered pairs in which the second number is the opposite of the first. (There should be $\underline{\overset{(7)}{?}}$ pairs.)
$[(^-3,^+3), (^-2,^+2), (^-1,^+1), (0,0), (^+1,^-1), (^+2,^-2), (^+3,^-3)]$
Graph this set in black on the same axes. (See TC 474)

6. Does the set of red points suggest a line? (yes) the green points? (yes) the blue points? (yes) the black? (yes)

BRAINTWISTERS

1. Graph the set of ordered pairs $\{A(^+1, ^+1)\ B(^+5, ^+1)\ C(^+3, ^+5)\}$ Draw $\overline{AB}$, $\overline{BC}$, $\overline{CA}$. Form new ordered pairs by doubling each number. Graph the ordered pairs and call the points D, E, and F. Draw $\overline{DE}$, $\overline{EF}$, $\overline{FD}$. Is $\triangle ABC$ congruent to $\triangle DEF$? (No) Does it have the same shape? (yes) Are corresponding angles congruent? (yes) (For graph, see TC 474)

2. Draw some other figure whose vertices have positive integers as coordinates. Find the coordinates of each vertex. Multiply each coordinate by 3 and draw the corresponding new figure. Are the figures the same shape? (Pupils figures will vary in shape. However each pupils figures should be the same shape but differ in size.)

3. Write a sentence that tells what you have observed from Exercises 1 and 2. (The figures are not congruent but are similar.)

VII. REFLECTIONS

Objective: To develop an understanding of geometric reflections.

Vocabulary: reflection, reflection in a vertical axis, reflection in a horizontal axis, reflection in a line

Materials: graph paper, pencil, straightedge, squared chalkboard

Teaching Procedure:

Use the chalkboard graph for the set of 3 ordered pairs given in Exercise 1 of Exploration. Follow the procedure as outlined in the pupil text. This will show a reflection of the figure in the vertical axis. The triangle DEF should be drawn on the chalkboard prior to the beginning of this Exploration.

A different figure than that given in the Exploration may be used. After this introduction ask the pupils to follow the Exploration in the pupils text. A small group that is having difficulty would profit by additional help from the teacher.

If the entire group finds this section difficult, then as a class, work out several examples similar to that of the Exploration.

The mathematical idea of reflection in a vertical axis is given at the end of Exercise 1. The meaning of reflection in a horizontal axis is brought out by Exercise 2. Exercise 1 and Exercise 2 are designed to help children understand the mathematics at work in reflections.

Exercise 2 of Braintwister gives many ordered pairs. If the pairs are graphed correctly and the segments drawn as indicated, the figure will roughly resemble an Indian. Children might enjoy making similar exercises for their classmates to graph.

REFLECTIONS

Exploration

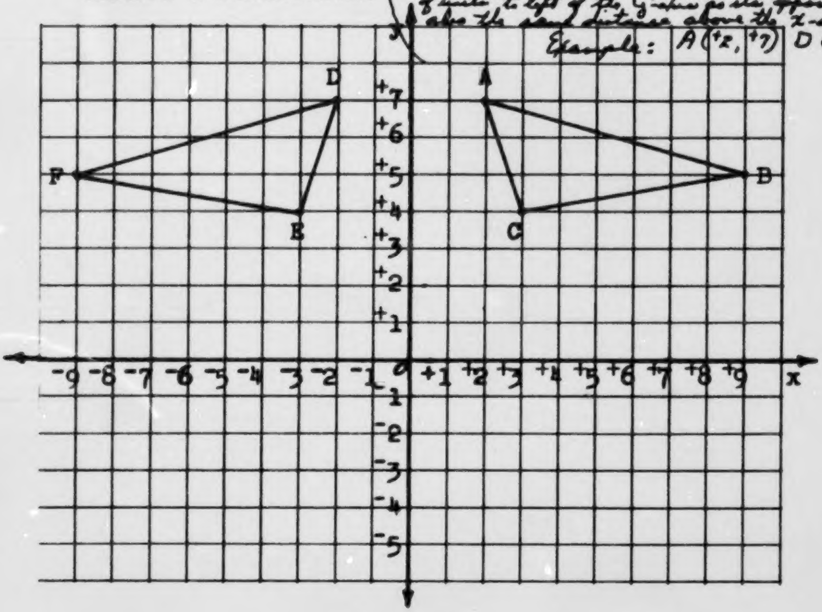
You already know at least one meaning for "reflection." We think of a mirror or a pool of clear water as giving a reflection. Let us see what reflections are in geometry.

1. a) Graph this set of ordered pairs:

{A($^+2$, $^+7$), B($^+9$, $^+5$), C($^+3$, $^+4$)}

Draw segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The union of these segments is a ___?___. (triangle)

Does your triangle ABC look like triangle ABC below? (Yes, it should.) This drawing shows also triangle DFE which is a reflection of triangle ABC. Do you see why it is called a reflection? (Yes, point D is opposite A; point E is opposite C; point F is opposite B. Each point is the same number of units to left of the y-axis as its opposite, and also the same distance above the x-axis. Example: A($^+2$, $^+7$) D($^-2$, $^+7$))



b) Copy the table below and write the coordinates of points D, E, and F. Fill in the distance of each point from the y-axis also.

Point	Coordinates of point	Distance of point from y-axis
A	($^+2$, $^+7$)	___?___ (2 units)
D	($^-2$, $^+7$)	___?___ (2 units)
B	($^+9$, $^+5$)	___?___ (9 units)
F	($^-9$, $^+5$)	___?___ (9 units)
C	($^+3$, $^+4$)	___?___ (3 units)
E	($^-3$, $^+4$)	___?___ (3 units)

c) What do you observe about the coordinates of points A and D? (The y-coordinate is the same. A is 2 units to the right of the y-axis; D is 2 units to the left of the y-axis.)

d) What do you observe about the distances from A to the y-axis and from D to the y-axis? (A is 2 units to the right of the y-axis. D is 2 units to the left of the y-axis.)

e) Are the observations you made for the points A and D similar for B and F? (yes) for C and E? (yes)

f) Mark and label points D, F, and E on your graph. Draw triangle DFE.

g) Fold your paper along the y-axis and hold your paper up to the light. Does A fall on D? (yes) Does B fall on F? (yes) C on E? (yes)

h) Is $\triangle ABC \cong \triangle DFE$? (yes)

Point D is a reflection of point A in the y-axis.

Point F is a reflection of point B in the y-axis.

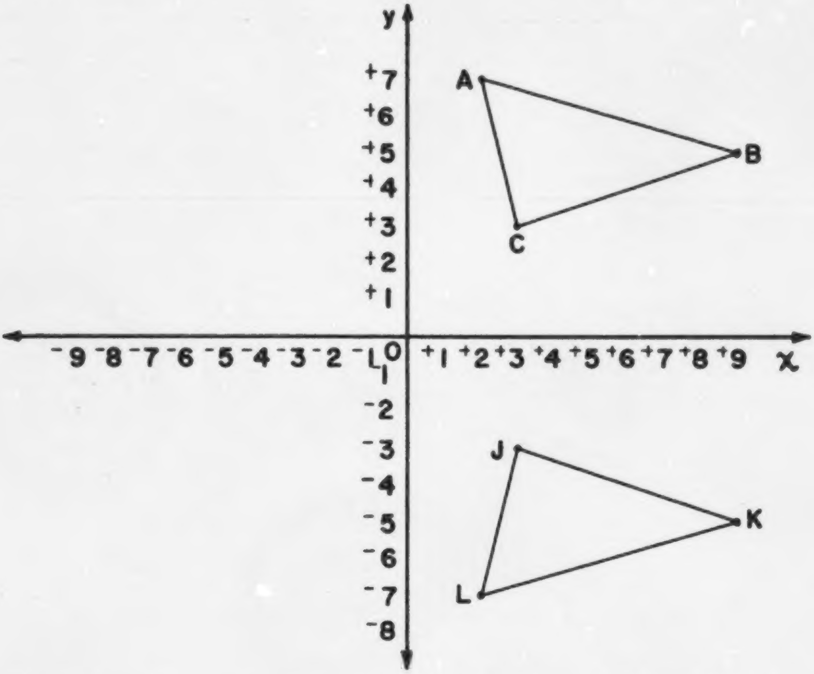
Point E is a reflection of point C in the y-axis.

$\triangle DFE$ is a reflection of $\triangle ABC$ in the vertical axis.

Note that we get a reflection of a point in the vertical axis when the first coordinate of the point is replaced by its opposite, and second coordinates remain the same.

A figure is a reflection of another in the vertical axis if corresponding points are the same distance from the vertical axis but in opposite directions from it.

2. How can we get a reflection of $\triangle ABC$ in the horizontal axis? In the drawing below, does triangle LKJ look like a reflection of triangle ABC in the horizontal axis? (yes)


y
+7
+6
+5
+4
+3
+2
+1
A
B
C
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9
x
-1
-2
-3
-4
-5
-6
-7
-8
J
K
L

a) Copy the table below and fill in the missing facts.

Point	Coordinates of point	Distance from x-axis
A	(+2, +7)	___?___ (7 units)
L	(+2, -7)	___?___ (7 units)
B	(+9, +5)	___?___ (5 units)
K	(+9, -5)	___?___ (5 units)
C	(+3, +3)	___?___ (3 units)
J	(+3, -3)	___?___ (3 units)

b) What do you observe about the coordinates of:

(1) A and L ? (The x-coordinate is the same. A is 7 units above the x-axis L is 7 units below the x-axis)

(2) B and K ? (The x-coordinate is the same. B is 5 units above the x-axis and K is 5 units below the x-axis.)

(3) C and J ? (The x-coordinate is the same. C is 3 units above the x-axis and J is 3 units below the x-axis.)

c) What do you observe about the distance of:

(1) A and L from the x-axis? (7 units)

(2) B and K from the x-axis? (5 units)

(3) C and J from the x-axis? (3 units)

d) How do these observations compare with those you made from the table in Exercise 1? (The y-coordinates are the same in Ex. 1. Here the x-coordinates are the same.)

e) Along which axis would you fold this drawing so that A, B, and C would fall on the corresponding points L, K, and J? (Fold along the x-axis.)

f) When the second coordinate of each point is replaced by its opposite and the first coordinate remains the same, we get a reflection in the ___?___ (x) axis.

A figure is a reflection of another figure in the horizontal axis if corresponding points are the same distance from the horizontal axis, but in opposite directions from it.

3. a) Graph this set of ordered pairs:

A($^-5$, $^-2$)   D($^-9$, $^-10$)   G($^-7$, $^-6$)
B($^-1$, $^-4$)   E($^-9$, $^-4$)   H($^-7$, $^-10$)
C($^-1$, $^-10$)   F($^-9$, $^-6$)   (Note: See graph at the bottom of TC 483.)

Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$, $\overline{FG}$, $\overline{GH}$.

This figure looks like a drawing of a ___?___ (house).

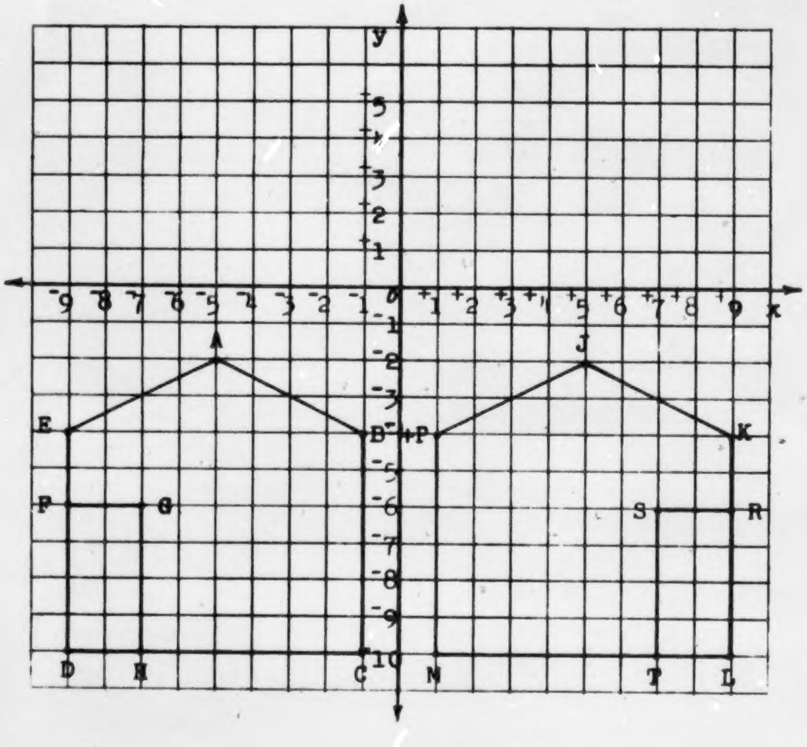
b) Graph the reflection of point A in the y-axis.

The reflection of A in the y-axis should have the same ? (y) coordinate as A. Its ? (x) coordinate should be the opposite of A's.

Point A and its reflection in the y-axis should be the same distance from the ? (y) axis. (Really they are also the same distance from the x-axis. We are trying to show the reflection in the y-axis when the x-coordinate is replaced by its opposite.)

c) Graph the reflections in the y-axis of the other labeled points. Draw segments to get the reflection of the figure.

d) Do your figures look like the ones in the graph below? (yes) Label the points of your reflection figure as shown.



e) Name pairs of points so that one point is the reflection of the other. Tell how far each point is from the y-axis. Pairs of points: A and J, 5 units; B and P, 1 unit; C and M, 1 unit; D and L, 9 units; E and K, 9 units; F and R, 9 units; G and S, 7 units; H and T, 7 units

f) Find a point on $\overline{AE}$ halfway between A and E. Call it W. What are the coordinates of W? $(^-7, ^-3)$

Find a point on $\overline{JK}$ halfway between J and K. Call it N. What are the coordinates of N? $(^+7, ^-3)$

Is N the reflection of W? (yes) Do you think all points of one figure are the reflections of corresponding points of the other? (yes)

g) Along what line can you fold your graph so that points of the reflection fall on corresponding points of the figure? (along the y-axis)

This is an example of reflection in a line. The line of reflection in this case is the y-axis.

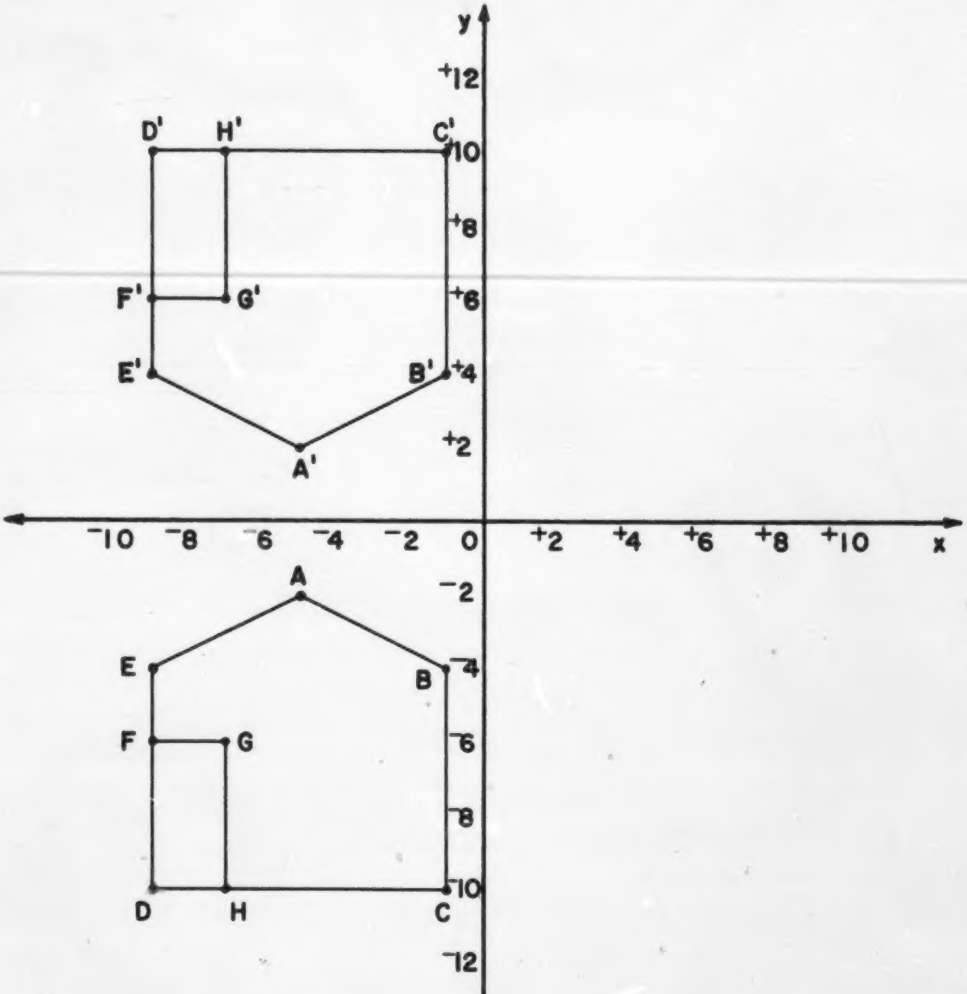
h) Draw the reflection of polygon ABCDE in the horizontal axis. (See TC 485)

(1) is this an example of reflection in a line? (yes) If so what line? (x-axis)

(2) Does the line of reflection always have to be the x-axis or the y-axis? (No)

Answers: Exploration

Exercise 3h


y
+12
+10
+8
+6
+4
+2
0
-2
-4
-6
-8
-10
-12
-10
-8
-6
-4
-2
+2
+4
+6
+8
+10
x
D'
H'
C'
F'
G'
E'
B'
A'
A
E
B
F
G
D
H
C

Exercise Set 8

1. a) (1) Graph this set of ordered pairs. Label each point with its letter and coordinates. (See TC 487)

   {A($^+6$, $^+4$), B($^+8$, $^+2$), C($^+6$, $^+1$)}

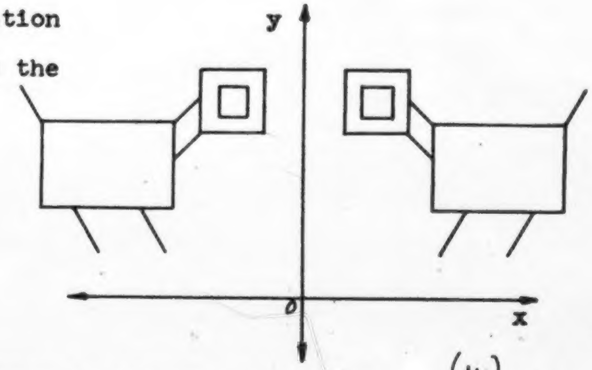
   (2) Draw $\overline{AB}$, $\overline{BC}$, $\overline{AC}$.

   (3) Is your figure a triangle? (yes)

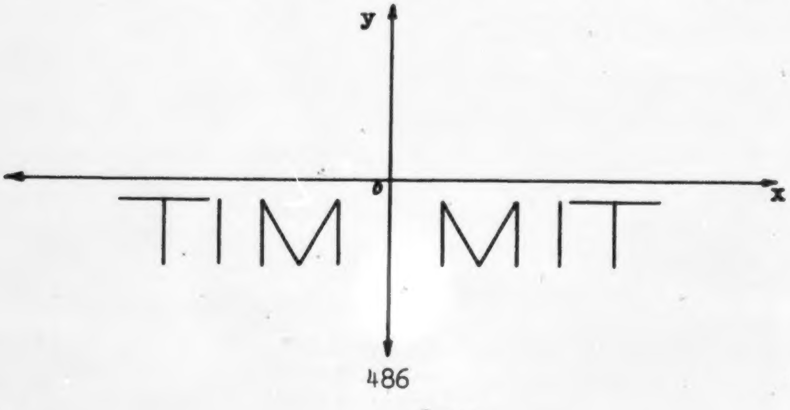
   b) Graph the reflection in the vertical axis of ΔABC. Label each vertex of this second triangle with its coordinates. (See TC 487)

   c) Graph the reflection in the horizontal axis of ΔABC. Label each vertex of this third triangle with its coordinates. (See TC 487)

2. The line of reflection in this drawing is the ___?___ (y) -axis.



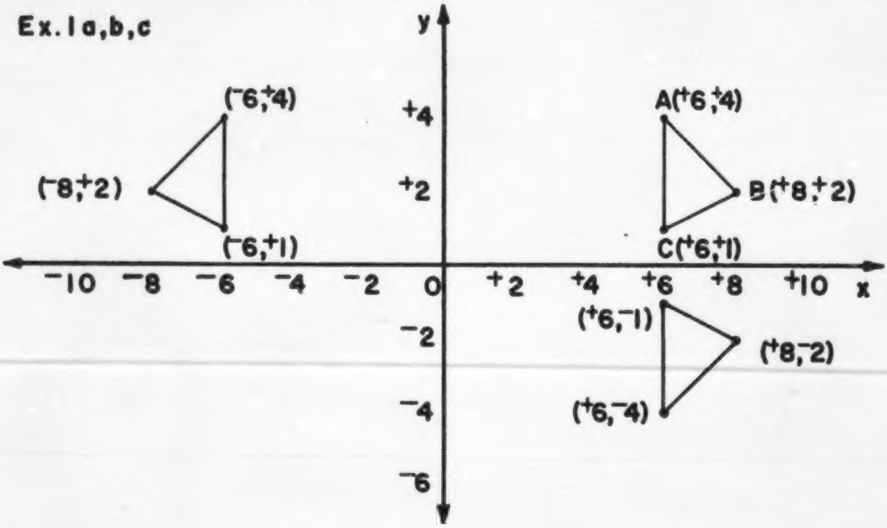
3. The line of reflection in this drawing is the ___?___ (y) -axis.



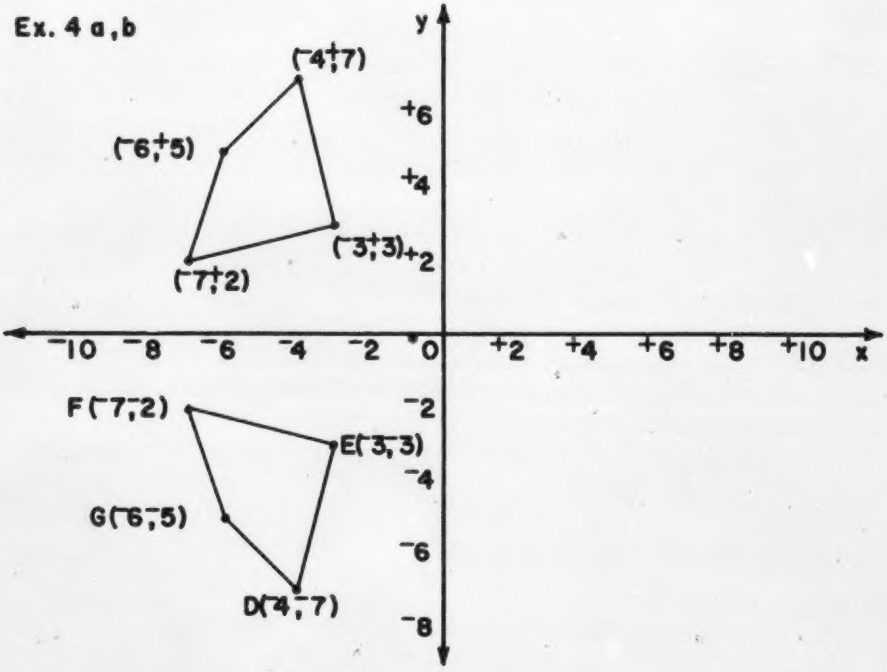
Answers: Exercise Set 8

Exercises 1a,b,c-4a,b

Ex.1a,b,c


y
x
(⁻6,⁺4)
(⁻8,⁺2)
(⁻6,⁺1)
A(⁺6,⁺4)
B(⁺8,⁺2)
C(⁺6,⁺1)
(⁺6,⁻1)
(⁺8,⁻2)
(⁺6,⁻4)
⁺4
⁺2
⁻2
⁻4
⁻6
⁻10
⁻8
⁻6
⁻4
⁻2
0
⁺2
⁺4
⁺6
⁺8
⁺10

Ex. 4 a,b


y
x
(⁻4,⁺7)
(⁻6,⁺5)
(⁻3,⁺3)
(⁻7,⁺2)
F(⁻7,⁻2)
E(⁻3,⁻3)
G(⁻6,⁻5)
D(⁻4,⁻7)
⁺6
⁺4
⁺2
⁻2
⁻4
⁻6
⁻8
⁻10
⁻8
⁻6
⁻4
⁻2
0
⁺2
⁺4
⁺6
⁺8
⁺10

4. a) Graph this set of ordered pairs: (See TC 487)

$\{D(^-4, ^-7), E(^-3, ^-3), F(^-7, ^-2), G(^-6, ^-5)\}$

Draw $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GD}$.

b) Graph the reflection of quadrilateral DEFG in the horizontal axis and label each vertex of this new quadrilateral with its coordinates. (See TC 487)

5. a) Draw the triangle whose vertices are points with coordinates $A(^+4, ^-3)$, $B(^-3, ^-2)$, $C(^-2, ^-10)$ (See TC 489)

b) Graph its reflection in the horizontal axis. Label each vertex of the new triangle with its coordinates. What is the line of reflection? (x-axis)

c) Graph the reflection of the triangle in Exercise 5a in the y-axis.

BRAINTWISTERS

1. a) Graph this set of ordered pairs. Label each point as you graph it.

$A(^+8, ^-2)$
$B(^+16, ^-10)$
$C(^+4, ^-13)$
$D(^+2, ^-13)$
$E(^+3, ^-9)$
$F(^+2, ^-9)$
$G(^+3, ^-6)$
$H(^+1, ^-6)$
$J(^+1, ^-5)$
$K(^+5, ^-5)$
$L(^+7, ^-5)$
$M(^+6, ^-4)$
$P(^+8, ^-8)$
$R(^+16, ^-1)$
$S(^+10, ^-4)$
$T(^+11, ^-5)$
$V(^+18, ^-3)$
$W(^+13, ^-7)$
$X(^+14, ^-8)$
$Y(^+20, ^-5)$
$Z(^+17, ^-11)$

(For graph, see TC 489)

b) Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$.

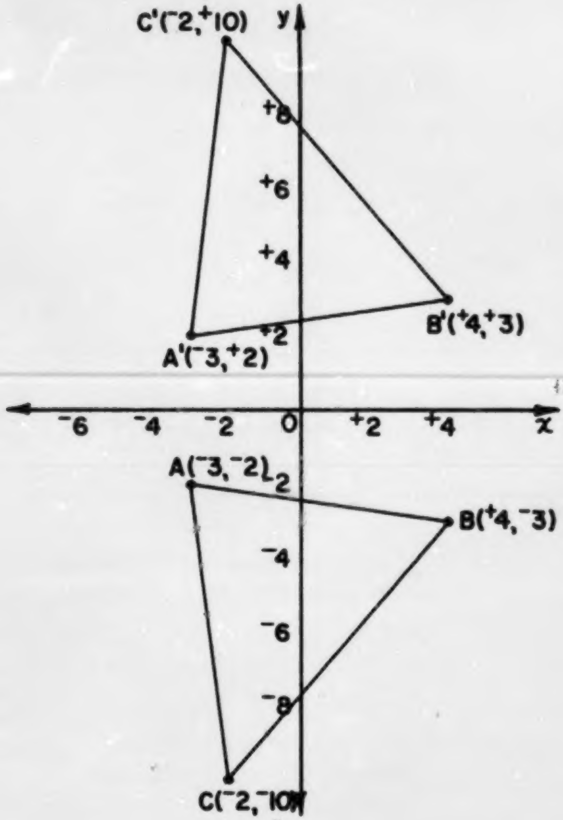
c) Draw $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HJ}$, and $\overline{JA}$.

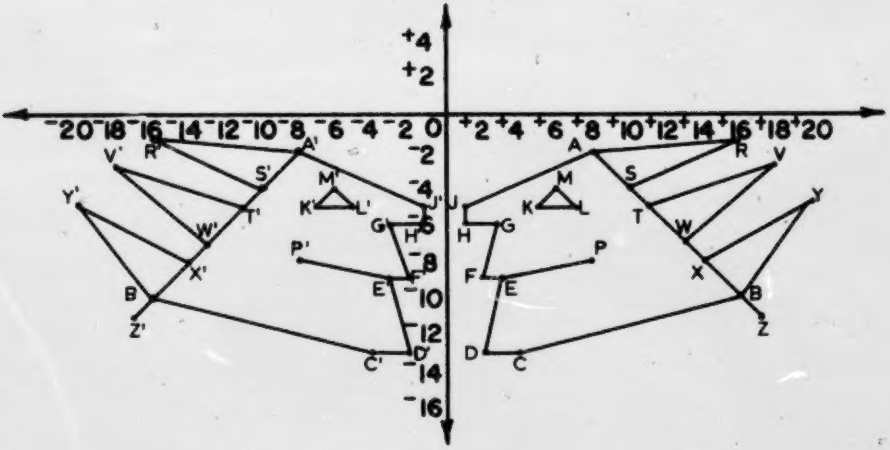
d) Draw $\overline{KL}$, $\overline{LM}$, $\overline{MK}$, $\overline{AR}$, $\overline{RS}$, and $\overline{EP}$.

e) Draw $\overline{TV}$, $\overline{VW}$, $\overline{XY}$, $\overline{YB}$, and $\overline{BZ}$.

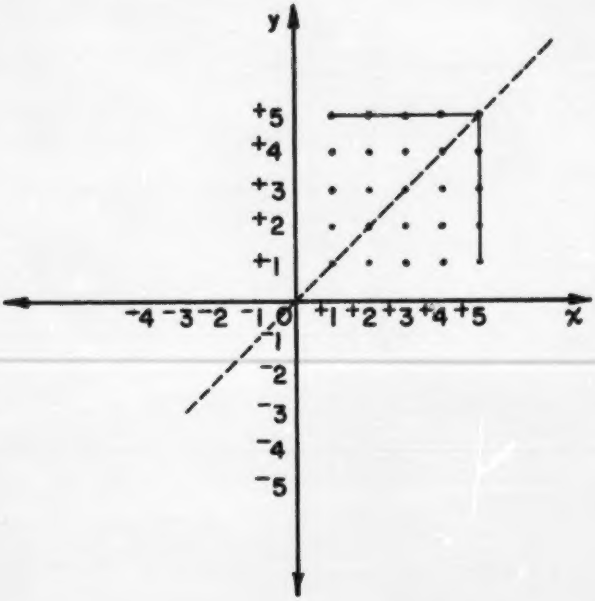
Answers: Exercise Set 8

Exercises 5a-5b-Braintwister 2


C'(⁻2,⁺10)
y
⁺8
⁺6
⁺4
⁺2
B'(⁺4,⁺3)
A'(⁻3,⁺2)
⁻6
⁻4
⁻2
0
⁺2
⁺4
x
A(⁻3,⁻2)
⁻2
B(⁺4,⁻3)
⁻4
⁻6
⁻8
C(⁻2,⁻10)

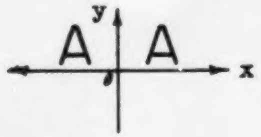

⁺4
⁺2
⁻20 ⁻18 ⁻16 ⁻14 ⁻12 ⁻10 ⁻8 ⁻6 ⁻4 ⁻2 0 ⁺2 ⁺4 ⁺6 ⁺8 ⁺10 ⁺12 ⁺14 ⁺16 ⁺18 ⁺20
⁻2
⁻4
⁻6
⁻8
⁻10
⁻12
⁻14
⁻16
R'
V'
A'
S'
Y'
T'
M'
K'
L'
J'
W'
G'
H'
P'
X'
E'
F'
B
Z'
C'
D'
A
R
V
S
Y
M
K
L
J
T
H
G
W
P
X
F
E
B
Z
D
C

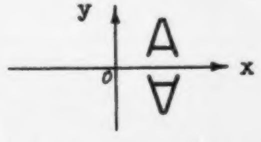
Answers: Braintwister 3


y
x
+5
+4
+3
+2
+1
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
-1
-2
-3
-4
-5

This is a drawing of Chief Pointed Head. Draw his reflection in the vertical axis. (See TC 489)

2. The reflection of the letter A in the vertical axis is also a letter A, but the reflection of the letter A in the horizontal axis is not.





Can you think of any other capital letters which would be the same as their reflections in the vertical axes? The horizontal axes? both axes?

(Vertical axis: H, I, M, O, T, U, V, W, X, Y, Z
Horizontal axis: B, C, D, E, H, I, K, O, X
Both axes: H, I, O, X
These answers are true if the letters are symmetrical.)

3. See the chart for Exploration Problem number 1 on page 280 in your text. Can you find an illustration of reflection of a set of points in a line which is not the x-axis and not the y-axis? (yes) If so, write the coordinates of a point and the coordinates of its reflection. [$(^+4, ^+5)$ and $(^+5, ^+4)$; $(^+3, ^+5)$ and $(^+5, ^+3)$; $(^+2, ^+5)$ and $(^+5, ^+2)$; $(^+1, ^+5)$ and $(^+5, ^+1)$] What do you observe? Test your observation on three other points. (The line of reflection is not the y-axis or the x-axis, but is half-way between these two axes, through the first and third quadrants) (See graph, TC 490)

4. Battleship Game # 3: The rules of this game are the same as those for Battleship Game # 1 (see end of Exercise Set 5) except that Rule d) is replaced by:

d)** (1) The first player calls out something like "reflection of $(^+1, ^+3)$ in the x-axis". If there is a point (battleship) marked with coordinates $(^+1, ^-3)$ then the marking there is erased (battleship sunk). If there is also a point marked at $(^+1, ^+3)$ it is erased (two battleships sunk at one firing!) If there is no point marked at $(^{+}1, {}^{-}3)$ (even though there might be one at $(^{+}1, {}^{+}3)$), then no markings are erased and the opponent then has his turn.

(2) If the first player calls out something like "reflection of $(^{+}1, {}^{+}3)$ in the y-axis" then the marking at $(^{-}1, {}^{+}3)$ is erased. If there is also a point marked at $(^{+}1, {}^{+}3)$, then that marking is also erased. If there is no point marked at $(^{-}1, {}^{+}3)$, then no markings are erased and the opponent has his turn.

5. Make up some rules for a harder Battleship game which illustrates the ideas of this chapter. Hint: see if you can use symmetry ideas to sink 3 or even 4 ships at once.

VIII. SYMMETRIC FIGURES

Objective: To develop an understanding of symmetric figures. To reinforce the ideas relating to reflection through the study of symmetry.

Materials: graph paper, squared chalkboard, scissors, pencil, straightedge, compass

Vocabulary: symmetric figure, line symmetry, line of symmetry, axis of symmetry

Teaching Procedure:

The Exploration and set of Exercises contained in this section are in sufficient detail in the Pupil Text to follow. Worthwhile review in graphing points of ordered pairs, review in constructing reflections, and stimulating challenge to the quick, able pupil are given in this section. Do not expect complete mastery by all pupils in the short time allotted for this section. Encourage each pupil to proceed as far as he can.

In the Exploration for this section, 1) is a very good activity which can be very easily combined with work in art. Some figures which can be made in this manner are valentines, snowflakes, Christmas trees and many others.

After all have had a chance to study the materials, discussion should center around the problems which pupils find puzzling or difficult. Encourage pupils to ask each other questions about the material. This crystallizes the ideas for those asking the question and is very meaningful for the pupils answering the question.

It is interesting to note the many examples of symmetry that one sees each day. Many of these are not just one figure, as for example a fireplace and mantle.

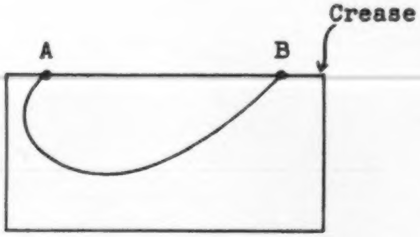
SYMMETRIC FIGURES

Exploration

1. Fold a piece of paper and crease the fold. Mark a point A and a point B on the crease.

   Start at A and draw a curve which does not intersect itself and ends at B.

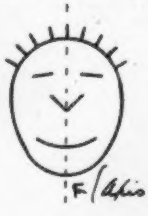
   Use scissors to cut along the curve. Be sure to cut through both parts of the sheet of paper. Unfold the part of the paper you cut out.



   The curve is a symmetric figure. The union of the curve and its interior is also a symmetric figure. Either set of points furnishes an example of line symmetry because when the paper is folded along the line suggested by the crease one part of the figure fits exactly on the other. The line represented by the crease is the line of symmetry or axis of symmetry.

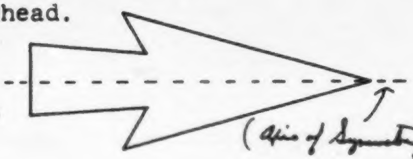
2. George has a "crewcut". Is this picture of his head a symmetric figure? (yes) If so, lay your ruler along the axis of symmetry.

(NOTE: Physiologists state that there really aren't any symmetrical heads or faces.)

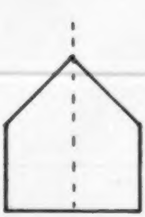


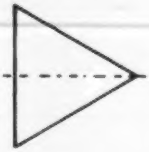
(Axis of symmetry)

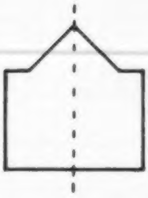
3. Chief Pointed Head made an arrowhead.
It looked like this:
Is it symmetric? If so, what is
the axis of symmetry?

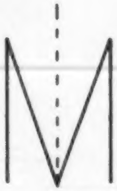

(Axis of Symmetry)

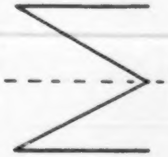
4. Trace these symmetric figures on a sheet of paper. Then use your ruler to draw the axis of symmetry on each of your tracings. (Axes of symmetry is shown as dotted lines.)

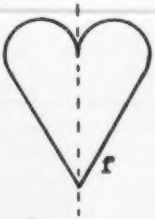

a

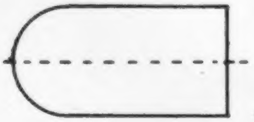

b

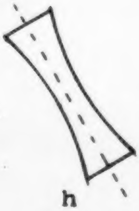

c

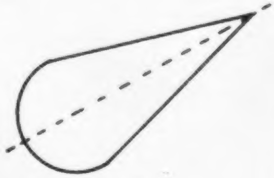

d

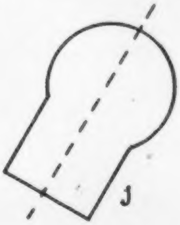

e


f


g


h


i


j

5. Must an axis of symmetry necessarily be either horizontal or vertical? (no)

6. Use your compass and straightedge to construct an equilateral triangle. Let the length of each side be three inches.

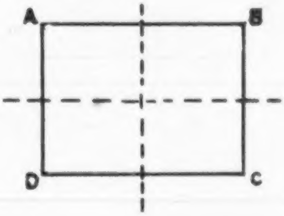
   Cut out the triangular region.

   a) Can you fold the paper to show an axis of symmetry? (yes) Can you show another axis of symmetry? (yes) How many axes of symmetry are there? (Three, an altitude drawn to each base.)

   b) Can a figure have more than one axis of symmetry? (yes) Must it have more than one? (No)

---

7. Trace this drawing of a rectangle.



   a) Is it a symmetric figure? (yes) If so, draw as many axes of symmetry as you can.

   b) How many axes of symmetry does a rectangle have? (two)

   c) How many axes of symmetry does a square have? (four)

8. Construct a circle with a radius of two inches. Cut out the circular region.

   a) How many axes of symmetry do you think a circle has? (An infinite number of axes. Any diameter is an axis of symmetry.)

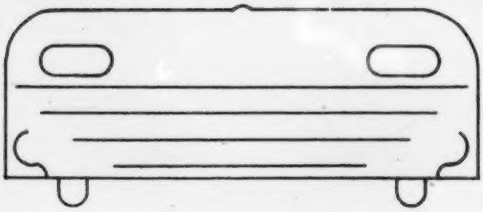
   b) What is the intersection of all the axes of symmetry of a circle? (the center of the circle.)

9. Are there examples of symmetric figures in your classroom? (Yes) If so, describe the axis of symmetry for each figure. (Answers will vary)

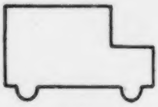
As you go home this afternoon look for examples of symmetric figures. Notice where the axis of symmetry is. You will find many examples. Look about your home for symmetric figures. Be especially alert at the dinner table in finding symmetric figures.

Exercise Set 9

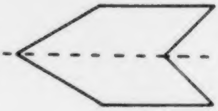
1. This is a drawing of a front of a wide bus. Is it a symmetric figure? (yes)



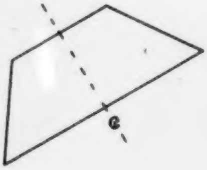
2. Which of these drawings seem to be symmetric figures? (b, c, d, e)



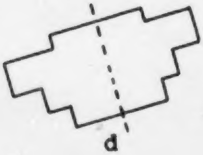
a



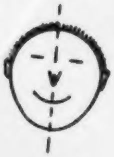
b



c

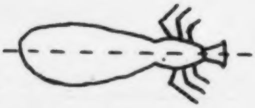


d



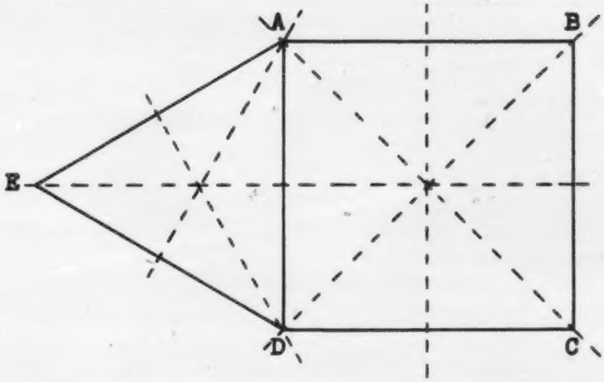
e

3. An insect might look like this. Does it appear to be a symmetric figure? (yes)



4.

Polygon ABCD is a square. Triangle ADE is an equilateral triangle.



Trace this figure on your paper.

a) Name three symmetric figures.
(Square ABCD, Triangle EAD, Polygon ABCDE)

b) Draw all of the axes of symmetry for square ABCD.
(See dotted lines in figure above. There are 4 axes of symmetry.)

c) Draw all of the axes of symmetry for equilateral triangle ADE. (See dotted lines in figure above. There are 3 axes.)

d) Is there an axis of symmetry for polygon ABCDE? (Yes, there is one.)

e) Look carefully at your drawing and the axes of symmetry you have drawn. If you consider these axes as well as the original figure, do there appear to be even more symmetric figures? (Yes)

5. Which of these are true statements?

   a) Axes of symmetry do not need to be vertical or horizontal. (True)

   b) A symmetric figure must have two axes of symmetry. (false)

   c) A figure is symmetric if half of it is a reflection of the other half. (true)

   d) An isosceles triangle is an example of a symmetric figure. (true)

6. Print the capital letters of the alphabet. Which are examples of symmetric figures?

(Horizontal axis of Symmetry: B, C, D, E, H, I, O, X, K,
Vertical axis of Symmetry: A, H, I, M, O, T, U, W, V, X, Y
Horizontal and Vertical Axis of Symmetry: H, I, O, X)

IX. SYMMETRY AND REFLECTION

Objective: To develop the understanding that the axis of symmetry can also be the line of reflection; to give further practice in graphing ordered pairs.

Materials: graph paper, pencil, squared chalkboard, straightedge

Vocabulary: No new words are used

Teaching Procedures:

The development in the Pupil Text is in sufficient detail to be used.

Braintwister 2 is quite long but should prove a challenge to some children. The pupil will be successful only if he follows the directions carefully.

SYMMETRY AND REFLECTION

Exploration

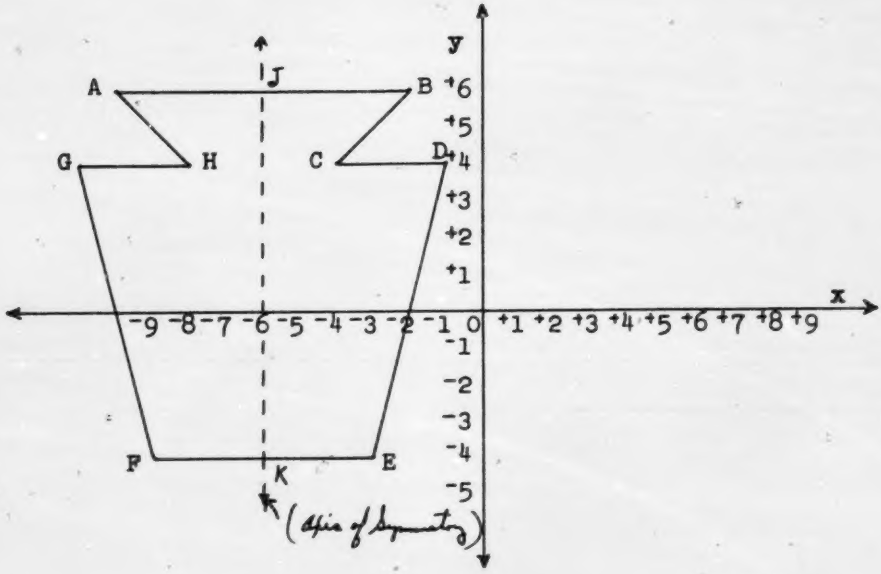
Is an axis of symmetry the same thing as a line of reflection? Think about this question as you work out these exercises.

1. a) Graph this set of ordered pairs.

$A(^-10, ^+6)$
$B(^-2, ^+6)$
$C(^-4, ^+4)$
$D(^-1, ^+4)$
$E(^-3, ^-4)$
$F(^-9, ^-4)$
$G(^-11, ^+4)$
$H(^-8, ^+4)$

Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{HA}$.

Does your drawing look like the one below? (Yes)


y
x
A
B
C
D
E
F
G
H
J
K
+6
+5
+4
+3
+2
+1
-1
-2
-3
-4
-5
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9
(Axis of Symmetry)

b) Is polygon ABCDEFGH a symmetric figure? (Yes)
Use your straightedge to draw the axis of symmetry.

c) Suppose you folded your paper on the axis of symmetry.
On what points would these points fall?
A on ___?___ (B) H on ___?___ (C) G on ___?___ (D) F on ___?___ (E)

d) Call the point where the axis of symmetry intersects $\overline{AB}$, point J. What are the coordinates of J? (-6, +6)

e) Do A, B, and J have the same y-coordinate? (yes) How long is $\overline{JB}$? (4 units) How long is $\overline{JA}$? (4 units)
Are A and B the same distance from the axis of symmetry? (yes)

f) Call the intersection of the axis of symmetry with $\overline{FE}$, point K. Are F and E the same distance from $\overleftrightarrow{JK}$? (yes)

g) Are H and C the same distance from $\overleftrightarrow{JK}$? (yes)
Are D and G the same distance from $\overleftrightarrow{JK}$? (yes)

h) What are the coordinates of the point of intersection of the x-axis with $\overline{GF}$? (-10,0) with $\overline{DE}$? (-2,0) with $\overleftrightarrow{JK}$? (-6,0)
Do these three points determine congruent segments on the x-axis? (yes)

Since corresponding points of the curve are the same distance from $\overleftrightarrow{JK}$, one part of the curve is the reflection of the other. $\overleftrightarrow{JK}$ is the line of reflection.

Curve JBCDEK is the reflection of curve JAHGFK.

Curve JAHGFK is the reflection of curve JBCDEK.

The axis of symmetry, $\overleftrightarrow{JK}$, is also the line of reflection.

2. a) Choose a point R in the interior of polygon ABCDEFGH whose coordinates are integers. (Do not choose a point on the axis of symmetry.) What are the coordinates of the point you chose? (Answers will vary.)

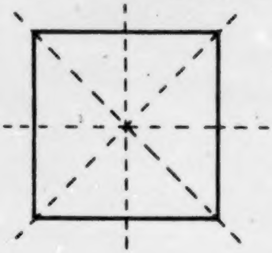
   b) Find the point which is the reflection of R in the axis of symmetry, $\overleftrightarrow{JK}$. Call it S. What are the coordinates of S? (The y-coordinate of S will be the same as the y-coordinate of R.)

   c) Find the point which is the reflection of R in the x-axis. Call it T. What are its coordinates? (The x-coordinates will be the same.)

   d) Find the point W which is the reflection of R in the y-axis. What are the coordinates of W? (The y-coordinates will be the same for W and K.)

Exercise Set 10

1. Trace this square on paper, and cut out the square and its region. How many axes of symmetry does it have? (four)



2. a) Graph this set of ordered pairs. (See TC 505)

   $A(^{+}3, ^{+}3)$, $B(^{+}3, ^{-}3)$, $C(^{-}3, ^{-}3)$, $D(^{-}3, ^{+}3)$.

   Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

   b) What kind of quadrilateral is polygon ABCD? (Square)

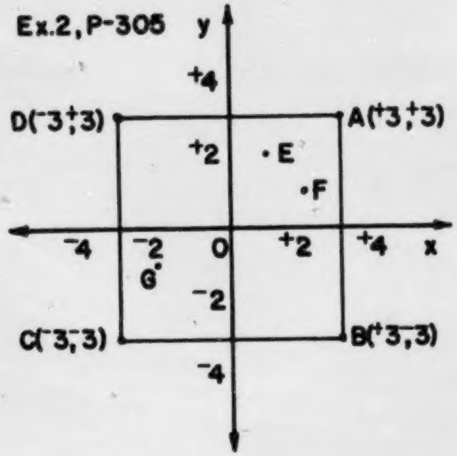
   c) The square ABCD has how many axes of symmetry? (four) Name them.
   ($\overline{AC}$, $\overline{DB}$, the x-axis, the y-axis)

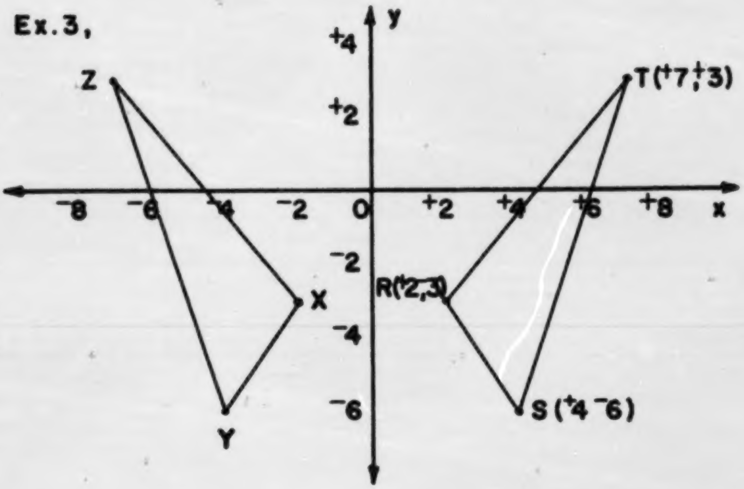
   d) What point is a reflection of A in the x-axis? (B) in the y-axis? (D) In $\overleftrightarrow{BD}$? (C)

   e) What point is the reflection of B in the x-axis? (A) in the y-axis? (C) In $\overleftrightarrow{AC}$? (D) In $\overleftrightarrow{BD}$? (B)

Answers: Exercise Set 10

Exercise 2 - Braintwister 1


Ex.2, P-305
y
$^{+}4$
D($^{-}3$,$^{+}3$)
A($^{+}3$,$^{+}3$)
$^{+}2$
E
F
$^{-}4$
$^{-}2$
0
$^{+}2$
$^{+}4$
x
G
$^{-}2$
C($^{-}3$,$^{-}3$)
B($^{+}3$,$^{-}3$)
$^{-}4$


Ex.3,
y
$^{+}4$
Z
T($^{+}7$,$^{+}3$)
$^{+}2$
$^{-}8$
$^{-}6$
$^{-}4$
$^{-}2$
0
$^{+}2$
$^{+}4$
$^{+}6$
$^{+}8$
x
$^{-}2$
R($^{+}2$,$^{-}3$)
X
$^{-}4$
$^{-}6$
S ($^{+}4$ $^{-}6$)
Y

3. a) Graph the set: $R(^{+}2, ^{-}3)$, $S(^{+}4, ^{-}6)$, $T(^{+}7, ^{+}3)$. Draw triangle RST. (See TC 505)

   b) Draw the reflection of triangle RST in the y-axis. Call the reflection triangle XYZ.

   c) Is the union of triangle RST and triangle XYZ a symmetric set of points? (yes) If so, what is the axis of symmetry? (y-axis)

BRAINTWISTERS

1. a) Look at your drawing for Exercise 2, Set 10. Graph the point $(^{+}1, ^{+}2)$, and label it E. (See TC 505)

   b) What point is the reflection of E in the x-axis? in the y-axis? in $\overleftrightarrow{AC}$ (call this reflection point F)? in $\overleftrightarrow{BD}$ (call this reflection point G)?

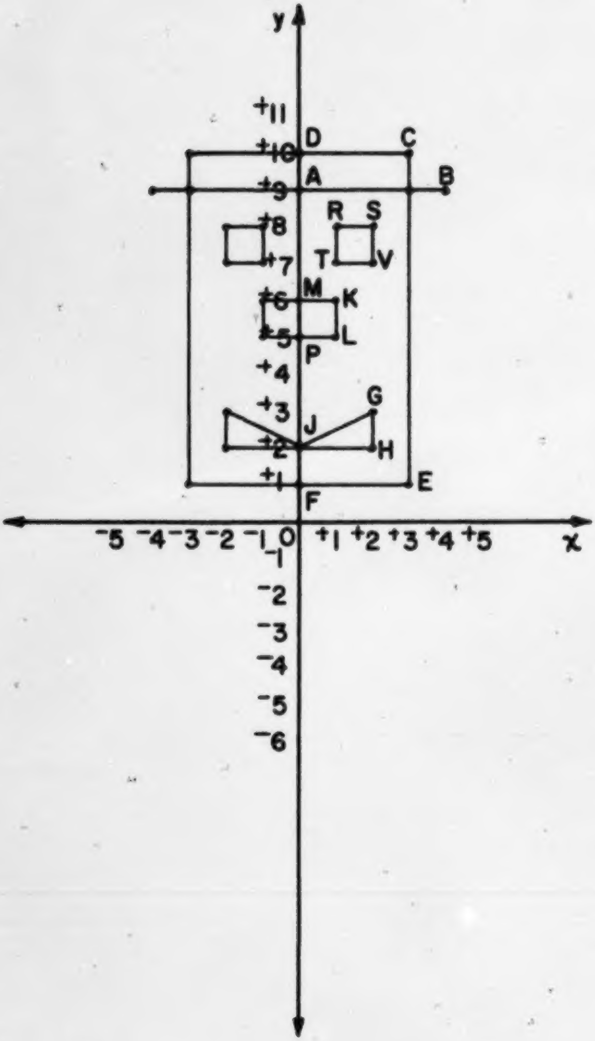
   c) What do you notice about the coordinates of these reflection points.

2. a) Graph this set of ordered pairs: (See TC 507)

   $A(0, ^{+}9)$
   $B(^{+}4, ^{+}9)$
   $C(^{+}3, ^{+}10)$
   $D(0, ^{+}10)$
   $E(^{+}3, ^{+}1)$
   $F(0, ^{+}1)$

   Draw $\overline{AB}$, $\overline{CD}$, $\overline{CE}$, $\overline{FE}$, $\overline{DF}$.

Answer: Braintwister 2


y
+11
D
C
+10
A
B
+9
R
S
+8
T
V
+7
M
K
+6
+5
L
P
+4
G
+3
J
+2
H
+1
E
F
−5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5
x
−1
−2
−3
−4
−5
−6

b) Graph these points using the same axes. (See TC 507)

G($^{+}2$, $^{+}3$)

H($^{+}2$, $^{+}2$)

J(0, $^{+}2$)

K($^{+}1$, $^{+}6$)

L($^{+}1$, $^{+}5$)

M(0, $^{+}6$)

P(0, $^{+}5$)

Draw $\overline{GH}$, $\overline{HJ}$, $\overline{GJ}$.

Draw $\overline{KL}$, $\overline{KM}$, $\overline{MP}$, $\overline{LP}$.

c) Graph this set of ordered pairs using the same axes.

R($^{+}1$, $^{+}8$)

S($^{+}2$, $^{+}8$)

T($^{+}1$, $^{+}7$) (See TC 507)

V($^{+}2$, $^{+}7$)

Draw $\overline{RS}$, $\overline{SV}$, $\overline{VT}$, and $\overline{TR}$.

d) Draw the reflection of the figure you now have in the vertical axis. (See TC 507)

e) You have a picture of Dandy Dan. Is it an example of a symmetric figure? (yes)